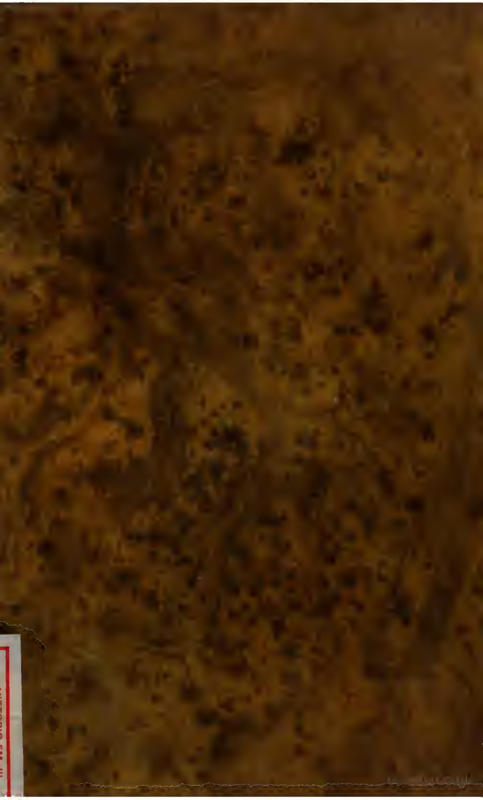

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ELEMENTS

OF THE

THEORY OF MECHANICS,

BY

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PROFESSOR OF MATHEMATICS IN THE UNIVERSITY
OF BOLOGNA.

TRANSLATED FROM THE ITALIAN,

By D. CRESSWELL, M.A.

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TO WHICH IS ADDED,

A SELECTION

OF

PROBLEMS IN MECHANICS.

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PREFACE

BY THE TRANSLATOR.

FOURTEEN years have now elapsed, since an anonymous critic, in a periodical publication of great celebrity, had the boldness to charge the members of the University of Cambridge, not only with an ignorance of the mathematical works of an illustrious foreigner, but with an incapacity to read and understand them, with any tolerable facility. And so general are the terms in which this censure is conveyed, that the writer does not except from it more than two or three of the most learned amongst us. Now, to any one who has been really acquainted with the University for the last twenty years, it is needless to point out the falshood of such an imputation. If, however, it had been asserted, that few attempts had then been made by us, to smooth the way of the student to a full comprehension of the abstruser parts of the *Mechanique Celeste*, it had been nearer to the truth. And, although much has since been done, and ably done, by some of our most eminent mathematicians, to remove that degree of blame, there is still, undoubtedly, room for further exertions of the same kind.

To the translator of the following work, a short treatise on Analytical Mechanics, has long appeared to be wanting; as a supplement to the books which are commonly read at this place, and as a key to a certain set of problems, which are very frequently proposed at our public Examinations. He had, therefore, made some collections, which might serve for an essay toward supplying that deficiency. But when the Mechanics of Venturoli fell in his way, he found the first part of the book so well adapted to the purpose, which he had in view, as to determine him to confine his labours to the mere office of translating and recommending it. The Italian author has, he thinks, shewn much good judgment in the management of his notation, in the selection, in the partition, and in the arrangement of his materials, as well as in the limits which he has given to them, with respect to quantity; and he is, in general, sufficiently happy in the clearness of his demonstrations. These, it will readily be allowed, are not the least considerable merits, in an elementary mathematical work. It would, indeed, have occasioned very little additional trouble to the translator, to have rendered the book still plainer than it is, by notes and illustrations. And, the author himself, after remarking that it is not easy to hold the just mean, between an excess of prolixity, and an excess of conciseness; and after doubting whether, in this particular, it be possible to satisfy the wants and the wishes of all readers, confesses that he has leaned rather toward the side of conciseness. But he assigns as his reason for so doing, that, in such a book as this, something ought to be left to the ingenuity and the diligence of the student. Fully agreeing with him in this opinion, the translator has wholly abstained from the business of annotation. All that he has added to the original, is a small selection of problems in Mechanics, intended for the exercise of the learner, and taken chiefly from the works of the earlier writers on that subject: and, for so doing, he might plead the authority of Newton himself, who has remarked, *in scientiis ediscendis, prosunt exempla magis quam præcepta.*

It only remains to apprize the English reader, that the following pages contain the first and the second book of the Mechanics of Venturoli. He treats of Mechanics, first, as a speculative, and then as a practical science. Taking up the notion of *body*, where Geometry leaves it, as that which has extension and figure, he further supposes it endued with the qualities of impenetrability, mobility, and inertia: he supposes it, also, to be acted upon by moving forces according to certain laws. This constitutes the first great division of his Work; and it will be found to comprise, in a short compass, the whole theory of Mechanics, which is thus reduced to a branch of pure mathematics. He next proceeds to shew, by an appeal to observation and

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experiment, that the supposed qualities and laws, which he had previously attributed to bodies, and to moving forces, are those which really obtain, in the physical world that surrounds us. And, lastly, he enters upon an application, of the mechanical principles, thus established, to researches relative to the equilibrium of structures and machines, the object of which, is practical utility.

But, however scientific this method may appear to be, yet considered as a method of teaching Mechanics, it seems liable to the objection, that the easiest and the most interesting parts of the subject, are those which it last presents to the attention of the learner. As, therefore, the substance of the greater portion of the last part of the work of Venturoli is sufficiently well taught in the books that are commonly in use amongst us, and as the first part is complete in itself, and is precisely that which appears to be wanted in the course of studies now pursued in the University, the translator has thought it best, for the present at least, to confine himself to the Theory of Equilibrium and of Motion.

Of the three general principles of Statics, namely, that of the composition of forces, that of the equilibrium of the lever, and that of virtual velocities, Venturoli has judiciously chosen the first, for the basis of his system. But of this he has given no other demonstration, but that which may be seen in Newton's Principia, and which must be understood to rest ultimately upon experiment. This is undoubtedly an omission. The fundamental proposition, however, concerning the parallelogram of forces, has been proved, independently of all but self-evident principles, both by Poisson and by Duchayla: and their demonstrations of it are too well known to stand in need of repetition in this book.

Trinity College, Cambridge,
June 1, 1822.

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Page 16. l. 2 from the top, *for* ΣP *read* $\Sigma . P$.

— l. 5 from the bottom, *for* $2x''^2$ *read* $2x'^2$.

17. l. 9 from the top, *for* ΣP *read* $\Sigma . P$.

22. l. 4 from the bottom, *for* involved *read* resolved.

33. l. 5 from the top, *for* there be *read* there will be.

56. l. 5 from the top, *for* k, k' *read* K, K'.

94. l. 14 from the top, *for* assymptote *read* asymptote.

111. l. 3 from the top, *after* the *insert* curve.

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141. l. 6 from the bottom, *for* $M'kl'$ *read* $M'k'l'$.

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— l. 6 from the bottom, *for* $\frac{v}{V}$ *read* $\frac{r}{V}$.

190. l. 14 from the bottom, *for* in the point *read* in the highest point.

ELEMENTS OF MECHANICS.

Book I.

ON EQUILIBRIUM.

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CHAP. I.

PRELIMINARY DEFINITIONS.

1. *MECHANICS* is the science of Equilibrium and of Motion.

That part of it, which relates to Equilibrium, is properly called *Statics*; that, which relates to Motion, is usually, although with less propriety, called *Dynamics*.

2. *Motion* is the passage from place to place; *Rest* is a permanency in the same place.

3. There is no motion without a determinate velocity and direction. *Velocity* is the ratio of the space, past over by whatever moves, to the time in which it is past over: it being here to be understood, that the comparison is not between the concrete quantities themselves, but between the numbers which represent them; that is, the ratios of each to its unity. *Direction* is the straight line past over by the moving body at any instant.

4. If the velocity is constant, the motion is said to be *uniform* or *equable*: if it changes, it is said to be *variable*.

And if the direction is constant, the motion is *rectilinear*; if it changes continually, it is *curvilinear*.

5. Motion and equilibrium are both of them properties of matter. By *matter* we further understand a substance endued with impenetrability and with inertia.

6. *Impenetrability* is that property by which different elementary particles of matter cannot, at the same time, occupy the same place.

7. *Inertia* is that property by which every elementary particle of matter, left to itself, preserves the state impressed upon it, whether it be of rest, or of uniform and rectilinear motion.

8. A *Body* is the aggregate of more elementary particles of matter than one, separated only by exceedingly small intervals.

The sum of the elementary particles, which constitute a body, is its *Mass :*

The intervals, which separate them, are called *Pores :*

The space occupied by the mass, and by the pores, is the *Volume :*

The ratio of the mass to the volume is the *Density* of a body.

9. *Force,* or *Power,* is the cause which impresses, or tends to impress, motion. Every force produces *actual* motion, if it be not counteracted by contrary forces. If it remain counteracted, the motion, which it tends to produce, is called *virtual.*

10. Forces cannot be known, nor measured, except by their effects. Now the effect of a force, applied to an element of matter, is to impress on it a certain velocity according to a certain direction. This velocity, whether it be actual or virtual, estimated in that direction, is assumed as the measure of the force.

11. *Equilibrium* is the state of any thing moveable acted upon by more forces than one, which mutually counteract each other.

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CHAP. II.

ON THE COMPOSITION OF FORCES.

12. *Proposition I.* If several forces act upon the same point, or elementary particle of matter, in any directions whatever, there will be compounded of these forces a single force equivalent to them.

For the motion, which the point will receive, or will tend to receive, by the simultaneous action of all the forces, must necessarily have a determinate velocity and direction; as if it were the effect of a single force.

13. The forces acting simultaneously on any moveable point, are called *components,* in respect of that single force which is equivalent to them, and which is called their *resultant.*

14. *Proposition II.* If two forces act upon a point at any angle, and if the parallelogram, having for sides the lines which represent them, be completed, its diagonal shall represent the resultant.

Let the point A (Fig. 1.) be acted upon by the two forces AB, AC; by the first of which, in a certain time (t) it would come to B describing the side AB; and by the second it would come to C, describing the side AC. Then, by the simultaneous action of the two forces, it will, in the time t, come to D, describing AD.

For the force AB, being parallel to CD, cannot cause the moveable point either to approach the side CD, or to recede from it. Wherefore, the point A, by the two forces, will arrive at the side CD, in the same time as it would arrive at it, by the single force AC, that is, in the time t. In a similar manner it may be proved, that the point A will arrive at the side BD in the time t. Therefore, at the end of the time t, the point must

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be found as well on the side CD as on the side BD. Wherefore, it will be found in D, the extremity of the diagonal, and (Art. 12.) will have described the diagonal. Therefore, &c.

15. *Coroll.* 1. The resultant of two *conspiring* forces is equal to their sum ; that of two *opposite* forces is equal to their difference.

16. *Coroll.* 2. In the triangle ABD, the sides AB, BD, AD, represent the three forces AB, AC, AD; and the sines of the angles BAD, BDA, ABD, are the same as the sines of the angles BAD, CAD, BAC, which are the angles made by the directions of the forces with each other. Thus, it is evident, that the triangle ABD represents, in its several parts, these three forces and their angles.

17. *Coroll.* 3. Each of the three forces AB, AC, AD, is proportional to the sine of the angle formed by the other two.

18. *Coroll.* 4. Any two of the three forces are to one another in the inverse ratio of the perpendiculars let fall on their directions, from a point taken at pleasure in the direction of the third.

19. *Proposition III.* If three forces meet in a point, with any directions that are not all in the same plane, and if the parallelepiped, having for sides the lines which represent them, be completed, its diagonal shall represent the resultant.

Let AM, AN, AO (Fig. 2.) be the three forces. The parallelepiped OM being completed, let there be drawn the diagonal AX, and the straight lines AY, OX.

The sides AO, XY being equal and parallel, so likewise will be AY, OX; and $AOXY$ will be a parallelogram having for its diagonal AX. Now (14.) AY will be the resultant of the two forces AM, AN; and AX will be the resultant of the two AY and AO, that is, of the three AM, AN, AO.

20. *Coroll.* In general, if more forces than one meet in a point, it is easy to assign the value and the position of the resultant.

For, describe a parallelogram, with sides which represent the first two forces, and its diagonal will be the resultant of the first two. Describe a new parallelogram, with this diagonal, and with the line, which expresses the third force, for its sides, and the new diagonal will be the resultant of the first three. Proceeding in this manner, the last diagonal will be the resultant of all the forces.

Or, describe a polygon, the sides of which, beginning from the point where the forces are applied, are successively equal and parallel to each of the forces, and taken in the same direction with each. Join the extremities of the first and last side of the polygon; and the straight line which thus shuts in the polygon shall be the resultant.

But if the forces and their directions are given in numbers, and if the resultant is to be similarly determined, the problem is solved by Trigonometry, as may be collected from what is said in Art. 16; by solving as many triangles, minus one, as there are forces.

21. *Proposition IV.* A given force may be resolved into two equivalent to it.

For, representing the given force by a straight line, and describing about it, as a diagonal, a parallelogram, the sides of the parallelogram will (14.) express two forces equivalent to it. The problem is, therefore, manifestly (Art. 16.) indeterminate; unless there be further given, either the values of the two component forces, or their directions; or, lastly, the value of one, and the direction either of that one, or of the other force.

23. The composition and resolution of forces are of continual use in Mechanics. It is, therefore, well to become familiar with the formulas, which express the relations between the resultant and the components, in the cases of most frequent occurrence. These formulas are all of them deduced from analytical Trigonometry; because the composition of forces and their resolution, under given conditions, are (16.) reducible to the solution of a triangle. They are investigated in the following Chapter.

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CHAP. III.

FORMULAS WHICH EXPRESS THE RESULTANT, WHEN THE COMPONENTS ARE GIVEN, AND CONVERSELY.

24. *PROPOSITION I.* IF two forces P and Q meet in a point, at an angle θ, to find the resultant S, and the angles α, β, which it makes with the components P, Q.

$$S = \sqrt{(P^2 + 2\,P\,.\,Q\,.\cos.\,\theta + Q^2)};$$

$$\sin.\,\alpha = \frac{Q}{S}\sin.\,\theta;\ \sin.\,\beta = \frac{P}{S}\sin.\,\theta.$$

25. *Coroll.* 1. If the two forces P and Q be equal,

$$\alpha = \beta = \tfrac{1}{2}\,\theta;\ S = 2\,P\cos.\,\alpha.$$

26. *Coroll.* 2. If the two forces P and Q meet at right angles,

$$S = \sqrt{(P^2 + Q^2)};$$

$$\sin.\,\alpha = \cos.\,\beta = \frac{Q}{S};\ \sin.\,\beta = \cos.\,\alpha = \frac{P}{S}.$$

27. *Coroll.* 3. If three forces P, Q, R, meet in a point at right angles, the resultant S, and the angles α, β, γ, which it makes with the three forces, will be determined as follows:

$$S = \sqrt{(P^2 + Q^2 + R^2)};$$

$$\cos.\,\alpha = \frac{P}{S};\ \cos.\,\beta = \frac{Q}{S};\ \cos.\,\gamma = \frac{R}{S}:$$

And hence, it manifestly follows, that

$$\cos.^2\,\alpha + \cos.^2\,\beta + \cos.^2\,\gamma = 1.$$

It is indeed known, from Geometry, that if a straight line make with three rectangular axes the three angles α, β, γ, the sum of the squares of their cosines is always equal to unity.

28. *Proposition II.* A force S being given, which makes with two axes, perpendicular to each other, and in the same plane

with it, the angles α, β, to resolve it into two P, Q, respectively parallel to the axes.

We shall have (26.) $P = S . \cos. \alpha$; $Q = S . \cos. \beta$; or, since $\cos. \beta = \sin. \alpha$, $P = S . \cos. \alpha$; $Q = S . \sin. \alpha$.

29. *Coroll.* In like manner, if the force S make with three rectangular axes the angles α, β, γ, and if it is to be resolved into three P, Q, R, parallel to the three axes, we shall have,

$$P = S . \cos. \alpha ; \quad Q = S . \cos. \beta ; \quad R = S . \cos. \gamma .$$

30. *Proposition III.* If the forces $S', S'', S''' \ldots\ldots$ meet in a point, and if the directions of these forces, referred to three rectangular axes, called the axes of x, of y, and of z, make with them respectively the angles α', β', γ'; $\alpha'', \beta'', \gamma''$; $\alpha''', \beta''', \gamma'''$; ..... to find the resultant S, and the angles α, β, γ, which it makes with the three axes.

Resolve (29.) each of the given forces into three, parallel to the three axes, and calling P the sum of those parallel to the axis of x, Q the sum of those parallel to that of y, R the sum of those parallel to that of z, we shall have, (15.)

$$A \begin{cases} P = S' \cos. \alpha' + S'' \cos. \alpha'' + S''' \cos. \alpha''' \ldots\ldots \\ Q = S' \cos. \beta' + S'' \cos. \beta'' + S''' \cos. \beta''' \ldots\ldots \\ R = S' \cos. \gamma' + S'' \cos. \gamma'' + S''' \cos. \gamma''' \ldots\ldots \end{cases}$$

Then compounding (27.) the three forces P, Q, R, there results

$$S = \sqrt{(P^2 + Q^2 + R^2)}$$

$$\cos. \alpha = \frac{P}{S}; \quad \cos. \beta = \frac{Q}{S}; \quad \cos. \gamma = \frac{R}{S}.$$

31. *Coroll.* 1. Here, also, in every set of three of the angles α', β', γ'; $\alpha'', \beta'', \gamma''$; $\alpha''', \beta''', \gamma'''$; &c. the following equations must (27.) obtain; viz.

$$\begin{aligned} \cos.^2 \alpha' + \cos.^2 \beta' + \cos.^2 \gamma' &= 1, \\ \cos.^2 \alpha'' + \cos.^2 \beta'' + \cos.^2 \gamma'' &= 1, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \cos.^2 \alpha + \cos.^2 \beta + \cos.^2 \gamma &= 1. \end{aligned}$$

32. *Coroll.* 2. If the forces meeting in a point be two only, S', S'', if the angle which they make with one another be θ, and if S be their resultant, we shall have, (24.)

$$S^2 = S'^2 + 2S'S'' \cos.\theta + S''^2;$$

Also (30.) $S^2 = P^2 + Q^2 + R^2 = (S' \cos.\alpha' + S'' \cos.\alpha'')^2 +$
$+ (S' \cos.\beta' + S'' \cos.\beta'')^2 + (S' \cos.\gamma' + S'' \cos.\gamma'')^2;$
whence, by expanding and reducing, since $\cos.^2\alpha' + \cos.^2\beta' + \cos.^2\gamma' = 1$, &c., there will result

$$S^2 = S'^2 + S''^2 + 2S'S''(\cos.\alpha' \cos.\alpha'' + \cos.\beta' \cos.\beta'' + \cos.\gamma' \cos.\gamma'').$$

Comparing this value of S^2 with the preceding, we shall have
$\cos.\theta = \cos.\alpha' \cos.\alpha'' + \cos.\beta' \cos.\beta'' + \cos.\gamma' \cos.\gamma''.$

By means of this theorem, knowing the inclination of two straight lines to three rectangular axes, we shall know also the inclination of the two straight lines to one another.

33. *Coroll.* 3. Conversely, let there be given the force S which makes the angles α, β, γ, with the co-ordinates x, y, z, and let it be required to resolve it into several forces.

First, resolve it (29.) into three P, Q, R, parallel to the three axes, and these will be $P = S\cos.\alpha$; $Q = S\cos.\beta$; $R = S\cos.\gamma$: each of which three forces may then be resolved into any number of forces, S', S'', S'''..... by means (30.) of the equations (A).

34. *Proposition IV.* On the curve AMB (Fig. 3.), referred to rectangular axes, if the point M be acted upon by a force P, in the direction of x, and by a force Q, in the direction of y, to substitute for these two forces, other two equivalent to them; the one (T) acting in the direction of the tangent MT, and the other (N) acting in the direction of the normal MN.

Considering the differential triangle of the curve, of which the hypotenuse $Mm = dz$, $Mr = dx$, $mr = dy$, it is plain, that, resolving the force P into two, one according to MT, the other according to MN, the first will (28.) be $P\cos.PMT = \frac{Pdx}{dz}$, the second $P\sin.PMT = \frac{Pdy}{dz}$. And similarly resolving the

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force Q into two, the one according to MT, the other according to MN produced, the first, $Q \cos. QMT = \frac{Q\,dy}{dz}$, the second, $Q \sin. QMT = \frac{Q\,dx}{dz}$.

Now the two component tangential forces conspire, and the two component normal forces oppose each other. Wherefore (15.),

$$T = \frac{P\,dx + Q\,dy}{dz}; \quad N = \frac{P\,dy - Q\,dx}{dz}.$$

If either of the forces P, Q, act in a direction opposite to the respective co-ordinates x, y, it must be taken negatively.

35. *Coröll.* Conversely, the two forces T, N, being given, the one acting according to the tangent MT, the other according to the normal MN, there may be substituted for them two other forces P, Q, acting in the direction of x, and of y. And

$$P = \frac{T\,dx + N\,dy}{dz}; \quad Q = \frac{T\,dy - N\,dx}{dz}.$$

The formula supposes that the directions of the forces T, N, make an acute angle with x taken positively. If either of these forces have a contrary direction, it must be considered as negative.

CHAP. IV.

ON THE COMPOSITION OF PARALLEL FORCES.

36. *Proposition.* If two parallel forces AP, BQ, (Fig. 4.) acting in the same direction, be applied to two points A, B, connected by an unalterable straight line AB, the resultant is parallel to the component forces, is equal to their sum, and divides AB into parts reciprocally proportional to the forces themselves.

Suppose two equal and opposite forces AM, BN, of any magnitude, to be applied to the points A, B, in the direction of AB produced; and, the parallelograms MP, NQ, having been completed, draw the diagonals AX, BY. Since the two additional forces AM, BN, mutually counteract each other, the resultant of the two forces AP, BQ, is the same as that of the four forces AP, AM, BQ, BN; or of the two AX, BY. Produce, therefore, XA, YB, until they meet in S; suppose the two forces AX, BY, to be transferred to S, and each of them to be resolved into two, the one according to KL, parallel to AB, the other according to SG, parallel to the forces AP, BQ. From this decomposition there will result (15.) according to KL, a force $= AM - BN = 0$: and according to SG a force $= AP + BQ$. Wherefore, the resultant of the two forces AP, BQ, is parallel to them, and is equal to their sum.

Further, $AP : PX :: SG : AG$;

and $QY : BQ :: GB : SG$;

also $PX = QY$;

$\therefore AP : BQ :: BG : AG$.

37. *Coroll.* 1. If the two forces AP, BQ, (Fig. 5.) act in directions opposite the one to the other, the resultant will be equal to their difference, it will act in the direction of the greater force, and the point G will no longer fall within the distance

AB between the two forces, but without it and on the side of the prevailing force; also, in this case, $AP : BQ :: BG : AG$.

38. *Coroll.* 2. Let there be a system of points A, B, C, D, (Fig. 6.) invariably connected with each other, and acted upon by the parallel forces P, Q, R, S; and let it be required to find their resultant. Join A, B, and divide AB in E in the inverse ratio of the forces P, Q. The resultant of the forces P, Q, will pass through E, and will be parallel to them and equal to $P + Q$. Similarly let EC, when drawn, be divided in F in the inverse ratio of the forces $P + Q$ and R. The resultant of the three forces P, Q, R, will pass through F, and will be parallel to them and equal to $P + Q + R$. Lastly, FD having been drawn, let it be divided in G, in the inverse ratio of the forces $P + Q + R$ and S. The resultant of the four forces P, Q, R, S, will pass through G, in the direction GX parallel to them, and will be equal to their sum. We may proceed in a similar manner for any number of forces.

If any one of the forces act in a direction opposite to that of the rest, it is necessary to operate as is indicated in Art. 37.

CHAP. V.

ON THE RESOLUTION OF A FORCE INTO OTHERS PARALLEL TO IT.

39. As of more parallel forces than one, one is compounded, so, by an inverse operation, a single given force may be resolved into more forces than one, parallel and equivalent to it. Where there is given only the number of the forces into which the proposed force is to be decompounded, the resolution may be made in an infinite number of ways, and the problem is indeterminate. It is rendered, however, determinate, by adding other conditions: of which the following problems will give an example.

40. *Proposition I.* To resolve the force S, applied to the point G (Fig. 4.) into two forces P, Q, parallel and equivalent to it, applied to the given points A, B, of the straight line AGB.

We shall have, by Art. 36,

$$S : P : Q :: AB : BG : AG.$$

41. *Coroll.* If the point G lie between the two points A, B, the two forces P, Q, will act in the same direction as the given force S, which will be equal to their sum: but if both A and B lie on the same side of G (Fig. 5.) the force P applied to the nearer point A, will conspire with the force S, that is, will act in the same direction, and the other force Q will act in a contrary direction; and the force S will be equal to the difference of the two P, Q.

42. *Proposition II.* To resolve the force S, applied to the point G (Fig. 7.) into three forces P, Q, R, parallel and equivalent to it, applied to the three given points A, B, C.

Join the given points; whence there results the triangle ABC, which, by the straight lines GA, GB, GC, is divided into the triangles AGB, AGC, BGC. Then,

$$S : P : Q : R :: ABC : BGC : AGC : AGB.$$

For produce AG to meet BC in E, and resolve (40.) the force S into two, P, X, equivalent and parallel to it, applied to the points A, E, and the force X, into two, Q, R, applied to the points B, C. Wherefore (40.),

$$S : P : X :: AE : GE : GA;$$

$$X : Q : R :: BC : CE : BE;$$

$$\therefore S : P : Q : R :: AE.BC : GE.BC : GA.CE : GA.BE.$$

But the four products $AE.BC$, $GE.BC$, $GA.CE$, $GA.BE$ are proportional to the four triangles ABC, BGC, AGC, AGB; because each of these triangles is represented by the corresponding product multiplied by the half of the sine of the angle AEB; wherefore

$$S : P : Q : R :: ABC : BGC : AGC : AGB.$$

43. *Coroll.* If the point G fall without the triangle ABC, the three forces P, Q, R, will not all act in the same direction as the force S. Let the point G fall out of the triangle, but within the angle ABC; the forces acting at B, C, will conspire with S, and the force acting at A will be opposed to it. The contrary will happen if the point G fall within the angle which is the vertical angle to A.

44. SCHOLIUM. If the three points A, B, C, lie in the same straight line, the problem is indeterminate; and the same thing happens if the given points be more than three: which we shall soon (57.) demonstrate.

CHAP. VI.

ON THE CENTRE OF PARALLEL FORCES.

45. ***Proposition I.*** In every system of points, of invariable form, acted upon by parallel forces, there is a certain point, called the *centre of parallel forces*, through which the resultant of the forces passes, whatever be the position of the system.

Let A, B, C, D (Fig. 6.) be a system of points acted upon by the parallel forces P, Q, R, S. Find (38.) the point G, through which the resultant GX passes. Then, however, the position of the system is altered, provided that the forces preserve their values, and remain parallel, the resultant will always still pass through the point G of the system. For, however the system revolve or incline, the point E will always cut AB in the inverse ratio of the forces P, Q; and the point F will always cut EC in the inverse ratio of $P + Q$ to R; and the point G will cut FD in the inverse ratio of $P + Q + R$ to S: whence (38.) the resultant will pass always through G.

46. *Coroll.* 1. Wherefore, the sum of all the forces may be understood to be concentered and collected in the centre of parallel forces; every system of parallel forces being thus reduced to a single point acted upon by one force only: and if this point be supported, the system remains in equilibrium in any position.

47. *Coroll.* 2. There are more methods than one of proceeding to find the centre of parallel forces. The first consists in the construction explained in Art. 38. The others consist in determining its position analytically, either with respect to three rectangular planes, or with respect to three given points; as will be shewn in the two following propositions.

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48. *Proposition II.* Any number of parallel forces, and the co-ordinates of their points of application being given, to determine the co-ordinates of the centre of the forces.

Let the points A, B, $C \ldots.$ be acted upon by the parallel forces P, Q, $R \ldots.$; and the system being referred to three rectangular axes, let x, y, z, x', y', z', x'', y'', z'',.... be the co-ordinates of those points: then shall the co-ordinates of the centre of parallel forces be

$$X = \frac{Px + Qx' + Rx'' \ldots}{P + Q + R}; \quad Y = \frac{Py + Qy' + Ry'' \ldots}{P + Q + R};$$

$$Z = \frac{Pz + Qz' + Rz'' \ldots}{P + Q + R}.$$

First, let there be only two points A and B (Fig. 8.) acted upon by the forces P, Q, and determined by the abscissas $OA' = x$, $OB' = x'$: let E be the centre of the two forces P, Q, and let the abscissa $OE = \xi$ correspond to it. Wherefore (36.) $P : Q :: BE : AE :: B'E' : A'E' :: x' - \xi : \xi - x.$

$$\therefore (P + Q)\,\xi = Px + Qx'.$$

Suppose now the point E and the third point C of the system to be joined, which is acted upon by the force R, and has, for its abscissa x'': if F be the centre of the two forces $P + Q$ and R, and if ξ' be its abscissa, it may be shewn, as before, that $(P + Q + R)\,\xi' = (P + Q)\,\xi + Rx''$; or, substituting the preceding value of ξ, $(P + Q + R)\,\xi' = Px + Qx' + Rx''$. Proceeding in the same manner, until we reach the centre of all the forces, to which the abscissa X corresponds, we shall have

$$(P + Q + R \ldots)\,X = Px + Qx' + Rx' \ldots..$$

Next, referring the position of the points successively to the axes of y and of z, we shall have, in a similar manner,

$$(P + Q + R \ldots.)\,Y = Py + Qy' + Ry'' \ldots..$$
$$(P + Q + R \ldots.)\,Z = Pz + Pz' + Pz'' \ldots..$$

from which equations are deduced the announced values of X, Y, Z.

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49. Scholium. These values may be written more concisely thus,

$$X = \frac{\Sigma . P x}{\Sigma P}; \; Y = \frac{\Sigma . P y}{\Sigma P}: \; Z = \frac{\Sigma . P z}{\Sigma . P}.$$

$\Sigma . P x$, &c. denoting the sum of the products of each force and its respective abscissa, and $\Sigma . P$ the sum of the forces.

50. *Coroll.* 1. If the points of the system lie all in one plane, and if the co-ordinates x, y, be taken on this plane, those denoted by z vanish, and there remain only the equations

$$X = \frac{\Sigma . P x}{\Sigma . P}, \; Y = \frac{\Sigma . P y}{\Sigma . P}.$$

And if the points of the system all of them lie in the same straight line, taking this line for the axis of x, y also vanishes, and there remains only the equation $X = \frac{\Sigma . P x}{\Sigma . P}$.

51. *Proposition III.* The distance, of the points A, B, C .... from a point O (Fig. 8.) and their distances from each other, being given, to determine the distance from O of the centre of the parallel forces acting at A, B, C,....

Let the distances $OA = d$, $OB = d'$, $OC = d''$.... ; $AB = f$, $AC = f'$, $BC = f''$.... ; and let D be the distance of the centre of the forces from O. Then,

$$D^2 = \frac{P d^2 + Q d''^2 + R d''^2 \ldots}{P + Q + R \ldots} - \frac{P Q f^2 + P R f'^2 + Q R f''^2}{(P + Q + R \ldots)^2}.$$

For (48.) $D^2 = X^2 + Y^2 + Z^2 =$

$$\frac{(P x + Q x' + R x'' \ldots)^2 + (P y + Q y' + R y'' \ldots)^2 + (P z + Q z' + R z'' \ldots)^2}{(P + Q + R \ldots)^2}.$$

But,

$$(P x + Q x' + R x'' \ldots)^2 = (P + Q + R \ldots)(P x^2 + Q x''^2 + R x''^2 \ldots)$$
$$- \{ P Q (x' - x)^2 + P R (x'' - x)^2 + Q R (z'' - x')^2 \ldots \}.$$

Likewise,

$$(P y + Q y' + R y'' \ldots)^2 = (P + Q + R \ldots)(P y^2 + Q y'^2 + R y''^2 \ldots)$$
$$- \{ P Q (y' - y)^2 + P R (y'' - y)^2 + Q R (y'' - y')^2 \ldots \};$$

and

$$(Pz+Qz'+Rz''\dots)^2=(P+Q+R\dots)\,(Pz^2+Qz'^2+Rz''^2)$$
$$-\ \{PQ\,(z'-z)^2+PR\,(z''-z)^2+QR\,(z''-z')^2\dots\}.$$

If now these squares be added together, and if their sum be substituted in the value of D^2, it is manifest, at a glance, since $x^2+y^2+z^2=d^2$, &c. and $(x'-x)^2+(y'-y)^2+(z'-z)^2=f'^2$, &c. that there results the value of D^2 above expressed.

52. Scholium. This value may be exhibited more concisely thus,

$$D^2=\frac{\Sigma.Pd^2}{\Sigma P}-\frac{\Sigma.PQf^2}{(\Sigma.P)^2},$$

$\Sigma.Pd^2$ denoting the sum of all the products which arise from multiplying each force by the square of its distance from O, and $\Sigma.PQf^2$ denoting the sum of all the products which arise from multiplying the products of the forces, taken two and two, by the squares of the distances of their several points of application.

53. *Coroll.* 1. If the position of the points A, B, C....be thus successively referred to three different fixt points, and if, by means of the preceding theorem, the distances D, D', D'' of the centre of forces from these three points be found, it is clear that the position of the centre itself will have been fully determined.

If the points of the system be all in one plane, it will be sufficient to refer them to two fixt points: and if they be all in one straight line, it will be sufficient to refer them to one fixt point.

These fixt points may be taken in the points of the system themselves, and then the position of the centre is determined by the mutual distances of the points of the system alone being given.

54. The *Momentum* of a force, with respect to any plane, is a name given to the product of the force itself and the distance of its point of application from that plane. And the plane, to which the momentums of all the forces of a system are referred, is called the *Plane of Momentums.*

The same word *Momentum* is sometimes used in a different sense, of which, when it happens, notice will be given.

55. *Proposition IV.* In every system of parallel forces, the momentum of the resultant, with respect to any plane, is equal to the sum of the momentums of the components.

For, with respect to the plane containing the axes of y, z, which may be any plane, the momentum of the resultant $= X \times \Sigma . P$; and the sum of the momentums of the components $= \Sigma . Px$; and (49.) these two quantities are equal to one another.

56. *Coroll.* 1. If the plane of momentums pass through the centre of the parallel forces, the sum of the momentums of the forces is nothing.

57. *Coroll.* 2. From what has been said are deduced the formulas, which determine the resultant of several parallel forces, when the components are given: and conversely.

Let the parallel forces be P, Q, $R \ldots$, and, a plane being supposed to pass across their directions, let $x, y, x', y', x'', y'' \ldots$ be the co-ordinates of the points of the plane, through which the forces P, Q, $R \ldots$ pass. Let S denote the resultant, and X, Y, the co-ordinates of that point of the plane through which it passes. We shall have the three equations,

$$\begin{aligned} S &= P + Q + R \ldots \\ (50.)\quad SX &= Px + Qx' + Rx'' \ldots \\ SY &= Py + Qy' + Ry'' \ldots \end{aligned}$$

Here, if the components be given, and the resultant is sought, it is evident at once that there are as many unknown quantities as there are equations, and the problem is therefore determinate. But if the resultant is given, and the components are sought, the problem is in its nature indeterminate.

If, however, the components are to pass through given points of the plane, the co-ordinates, x, y, and x', y', will be given, and there will remain to be determined only the components themselves P, Q, $R \ldots$ It is easily seen, then, that the problem becomes determinate when the given points are two; or when they are three, provided that they do not lie in the same straight line: which proves what we have asserted at the end of the preceding Chapter.

CHAP. VII.

ON THE CENTRE OF GRAVITY.

58. **Hypothesis.** Every elementary particle of matter is supposed to be continually acted upon by a force, by which it is urged to fall, in the direction of a straight line perpendicular to a plane which touches the surface of stagnant water. This force is considered as equal in all the particles of matter, and constant in each. Acted upon by it, they describe, whilst falling freely, in the first 1″ of their descent the space of 4.9044 metres.

59. The force, so acting, is called *Gravity;* the plane which touches the surface of stagnant water is called an *Horizontal Plane;* the plane perpendicular to this is called a *Vertical Plane.*

Verticals, not far distant from one another, may be considered as parallels, and therefore the directions of gravity, comprehended within short intervals, may be taken as parallels.

60. The *Weight*, or *Absolute Weight*, of a body is the resultant of all the forces of gravity which act upon each of its elements. This resultant is (38.) equal to their sum, and therefore the weight is proportional (8.) to the mass.

61. The *Specific Weight*, or *Specific Gravity*, of a body is the ratio of its absolute weight to its volume. Hence, the specific gravity is proportional to (8.) the density.

62. Let M be the mass of a body, V its volume, D its density, P its absolute weight, G its specific weight; and let g be the force of gravity. Then, (8.)

$$D = \frac{M}{V};\ (60.)\ P = Mg;\ (61.)\, G = \frac{P}{V};$$

whence, of the five indeterminate elements M, V, D, P, G, if two be known, the other three will be known.

The equations $M = \frac{P}{g}$, $P = GV$, of which the most frequent use is made in the solution of mechanical problems, ought more especially to be noted.

63. The *Centre of Gravity* of a body is the centre (45.) of the parallel forces of gravity, which act upon each of its elements. The weight of the body may (46.) be understood to be united and collected in the centre of gravity. In like manner, the centre of gravity of a system of bodies, is the centre of the weights of the bodies composing the system. Every one of these weights must be understood to be collected in the centre of gravity of the respective body.

64. *Coroll.* The centre of gravity may be found experimentally, by suspending the heavy body, or any model of it, by two different points in it. The intersection of the verticals which pass through the points of suspension will be the centre of gravity. It may be found geometrically, by the methods laid down, in Arts. 38.48.51, for finding the centre of parallel forces.

It is usual, also, to investigate the centre of gravity of lines and geometrical figures, upon the supposition that their elements are heavy and homogeneous. This research will form the subject of the three following Chapters.

CHAP. VIII.

INVESTIGATION OF THE CENTRE OF GRAVITY IN LINES, AND THE MORE SIMPLE FIGURES.

65. ***Proposition I.*** Lines, or figures, which are symmetrical about a point, an axis, a plane, have (36.) their centre of gravity in that point, in that axis, in that plane.

66. *Proposition II.* The centre of gravity of a triangle is in the straight line, drawn from any angle to the bisection of the opposite side, at the distance of two-thirds of that line from the angle.

Let ABC (Fig. 9.) be the triangle: bisect any two of its sides BC, AC, in D and E, and from the opposite angles A and B, draw the straight lines AD, BE. If the triangle be divided into elementary trapeziums by straight lines parallel to BC, it is plain that AD, which bisects each of these straight lines, will pass through the centres of gravity of all the elements of the triangle, and therefore also through the centre of gravity of the triangle itself. For reasons of the same kind, the straight line BE will also pass through the centre of gravity of the triangle. Wherefore this centre is in the intersection G of AD, BE.

Again, D, E being joined, DE cuts the sides BC, AC proportionally; wherefore, it is parallel to AB, and the triangles AGB, DGE, are similar;

$$\therefore AG : GD :: AB : DE :: BC : DC.$$

But, by the construction, BC is the double of DC; therefore AG is the double of GD, and consequently $AG = \frac{2}{3} AD$.

67. *Coroll.* By the two preceding propositions may be found the centre of gravity of the perimeter, or of the surface, of any polygon; by using the method of Art. 38, or that of Art. 48.

68. *Proposition III.* The centre of gravity of a triangular pyramid is in the straight line drawn from the vertex to the centre of gravity of the base, at the distance of three-fourths of that line from the vertex.

Let $V-ABC$ (Fig. 10.) be the pyramid; draw AD from A to the bisection D of BC, and take $AE=\frac{2}{3}AD$; E will (66.) be the centre of gravity of the triangle ABC, the base of the pyramid. Join V, D, and take $VF=\frac{2}{3}VD$, so that F is the centre of gravity of the triangular face VBC. Join V, E, and A, F. If now the pyramid be resolved into its constituent elements, by planes parallel to the base ABC, it is manifest that VE will pass through the centres of gravity of all these elements. Wherefore the centre of gravity of the pyramid must lie in VE: for reasons of the same kind, it must fall in AF. It will therefore fall in the intersection G of VE and AF.

Again, E, F being joined, EF cuts the lines AD, VD proportionally, and is therefore parallel to VA, and the triangles AGV, FGE, are similar;

$$\therefore VG:GE::VA:FE::AD:DE.$$

But, by the construction, AD is the triple of DE; therefore, VG is the triple of GE, and $VG=\frac{3}{4}VE$.

69. *Coroll.* 1. The proposition is true of *any* pyramid, and also of any cone. For if a straight line be drawn from the vertex of any pyramid to the centre of gravity of the base, it is manifest that the centre of gravity of the pyramid must be found in that straight line, because it passes through the centres of gravity of the elements, infinite in number, into which the pyramid is involved, by planes parallel to the base. Suppose now the base of the pyramid to be divided into triangles, by its diagonals, and the pyramid itself into as many triangular pyramids, by planes drawn through the vertex and through the diagonals. If at a

distance from the vertex equal to three-fourths of the altitude of the solid, a plane be drawn parallel to the base, it will pass (68.) through all the centres of gravity of the triangular pyramids: it must, therefore, pass through the centre of gravity of the given pyramid, which will consequently be found at the distance of three-fourths of the straight line drawn from the vertex to the centre of gravity of the base. And this is true, whatever be the number of faces of the pyramid, and is therefore true of a pyramid with an infinite number of faces, that is, of a cone.

70. *Coroll.* 2. Hence may be found the centre of gravity of any polyhedron; since every polyhedron may be resolved into pyramids.

CHAP. IX.

FORMULAS FOR THE CENTRES OF GRAVITY OF LINES AND PLANES.

71. *Proposition I.* For the centre of gravity of a curve line s, the co-ordinates of which are x and y,

$$X = \frac{\int x\,ds}{s};\ Y = \frac{\int y\,ds}{s}.$$

For, in general (50.) $X = \frac{\Sigma . Px}{\Sigma . P}$; $Y = \frac{\Sigma . Py}{\Sigma . P}$.

And here the force P is proportional to ds, the element of the arc.

72. *Coroll.* 1. If the arc be symmetrical about the axis of x, the centre falls on this axis, $Y = 0$, and the equation $X = \frac{\int x\,ds}{s}$ suffices.

73. *Coroll.* 2. If the arc have a double curvature, it will be proper to refer it to three axes, and

$$X = \frac{\int x\,ds}{s};\ Y = \frac{\int y\,ds}{s};\ Z = \frac{\int z\,ds}{s}.$$

74. *Coroll.* 3. If these formulas be applied to find the centre of gravity of an arc of a circle, this centre is found in the radius which bisects the arc; and its distance from the centre of the circle is a fourth proportional to the arc, its chord, and the radius.

75. *Proposition II.* For the centre of gravity of a plane surface, $X = \frac{\int Xy\,dx}{\int y\,dx}$; $Y = \frac{\frac{1}{2}\int y^2\,dx}{\int y\,dx}$.

The demonstration is the same as that of Art. 71, observing that the element of the surface $= y\,dx$: and the first of these two equations will suffice, if the proposed surface be symmetrical about the axis of x.

76. *Coroll.* The following applications may serve for practice.

1. A trapezium, with parallel bases, has its centre of gravity in the straight line which bisects the bases. Let the two bases be p, q, and let a be the straight line joining their bisections. The distance in this line between the centre of gravity and the base p is $\frac{a}{3} \cdot \frac{p + 2q}{p + q}$.

2. A segment of a circle has its centre of gravity in the radius which bisects it; and its distance from the centre of the circle is $\frac{1}{12}$ of the cube of its chord divided by the area of the segment.

3. A sector of a circle has its centre of gravity in the radius which bisects it; and its distance from the centre of the circle is a fourth proportional to the arc, its chord, and $\frac{2}{3}$ of the radius.

4. The centre of gravity of a segment of an ellipse terminated by the axis minor, and by an ordinate to the axis major, is the same point as the centre of gravity of the corresponding segment of a circle described about the ellipse.

5. A common parabola has its centre of gravity in the axis, at a distance from the vertex of $\frac{3}{5}$ of the axis.

CHAP. X.

FORMULAS FOR THE CENTRES OF GRAVITY OF SURFACES, AND OF SOLIDS, GENERATED BY REVOLUTION.

77. *Proposition I.* The centre of gravity of a surface, generated by the revolution of a curve, is in the axis, at a distance X from the origin of the axis, such that $X = \frac{\int x\,y\,ds}{\int y\,ds}$.

This value is easily deduced, from Art. 50., it being observed that the element of the surface is $2\pi y\,ds$.

78. *Coroll.* The centre of gravity of the surface of a spherical cap is in the bisection of its sagitta.

79. *Proposition II.* The centre of gravity of a solid, generated by the revolution of a plane, is in the axis, at a distance X from the origin of the axis, such that $X = \frac{\int x y^2 dx}{\int y^2 dx}$.

This value is easily obtained, it being observed that the element of the solid is $\pi y^2 dx$.

80. *Coroll.* From the application of this formula to the investigation of the centres of gravity of various solids, we have the following results; the abscissas being taken from the vertex of the figure which, by its revolution, generates the solid.

1. For a spherical segment, the radius being a,

$$X = \frac{8a - 3x}{12a - 4x} \cdot x.$$

2. For a spherical sector,

$$X = \tfrac{1}{8}(2a + 3x).$$

3. For a hemisphere,

$$X = \tfrac{5}{8}a.$$

4. For a segment of a paraboloid,

$$X = \tfrac{2}{3}x.$$

5. For a segment of an ellipsoid; if a be the semi-axis about which the rotation takes place, then, as in a spherical segment,

$$X = \frac{8a - 3x}{12a - 4x} \cdot x.$$

6. For a segment of an hyperboloid,

$$X = \frac{8a + 3x}{12a + 4x} \cdot x.$$

This value is always contained between $\frac{2}{3}$ and $\frac{3}{4}$ of the abscissa x.

7. For the surface of a cone; if a be its axis,

$$X = \tfrac{2}{3}a.$$

CHAP. XI.

ON THE USE OF THE CENTRE OF GRAVITY IN MEASURING SURFACES AND SOLIDS GENERATED BY REVOLUTION.

81. ***Proposition I.*** The surface generated by the revolution of an arc is equal to the arc itself multiplied by the path of its centre of gravity.

Let Y be the radius of the circumference described by the centre of gravity; hence its path $= 2\pi Y$. But (71.) $Y = \frac{\int y\,ds}{s}$. Wherefore, the path of the centre of gravity of the arc multiplied by the arc iself $= 2\pi \int y\,ds$; which is the well known value of the surface generated by the revolution of the arc s about the axis of x.

82. *Coroll.* If several arcs revolve about a common axis, the sum, or the difference, of the surfaces generated by them, is equal to the sum, or the difference, of the arcs themselves, multiplied by the path of their common centre of gravity.

Let the arcs be $s, s', s'' \ldots.$ and the ordinates from their centres of gravity, $v, v', v'' \ldots.$ The surfaces generated will be (81.) $2\pi s v$, $2\pi s' v' \ldots.$ and their sum will be $2\pi (sv + s'v' \ldots.)$.

Also the common centre of gravity of all the arcs will have (48.) for its ordinate, $\frac{sv + s'v' + s''v'' \ldots.}{s + s' + s'' \ldots.}$; and multiplying the path of this common centre by the sum, $s + s' + s'' \ldots$, of the arcs themselves, there will result $2\pi (sv + s'v' + s''v'' \ldots)$. Wherefore, &c.

The same reasoning applies to the difference of the arcs.

83. *Proposition II.* The solid generated by the revolution of a plane surface is equal to the surface itself multiplied by the path of its centre of gravity.

For (75.) $Y = \frac{\frac{1}{2}\int y^2\,dx}{\int y\,dx}$. Wherefore, the surface multiplied by the path of the centre of gravity $= \pi \int y^2\,dx$, the well known expression for the generated solid, π being the circumference of a circle the diameter of which is unity, and y being the diameter of the section of the solid at the distance x.

84. *Coroll.* If several surfaces revolve about the same axis, the sum, or the difference, of the solids, generated by them, is equal to the sum, or the difference, of the surfaces themselves multiplied by the path of their common centre of gravity.

85. *Proposition III.* If a straight line move so as to remain always perpendicular to the line described by its centre of gravity, the surface generated by the motion of the straight line is equal to the straight line itself multiplied by the path of its centre of gravity: and the same is true in the case of a plane, which, moving in like manner, generates a solid.

For every element of the surface, or of the solid, generated will be equal to the product of the generating straight line, or surface, by the corresponding element of the line described by the centre of gravity.

CHAP. XII.

METHOD OF FINDING BY APPROXIMATION THE SURFACES, THE SOLID CONTENTS, AND THE CENTRES OF GRAVITY, OF FIGURES WHICH ARE NOT DEFINED BY EQUATIONS.

86. WHENEVER the equation of a proposed curve is not given, neither the value of the figures bounded by the curve, or generated by its revolution, nor the position of their centres of gravity, can be obtained otherwise than in the way of approximation. The most obvious method of approximation consists in assuming a great number of ordinates at small and equal distances, and in then considering the small arcs included by them as rectilinear; thus transforming the curve into a polygon. Greater accuracy, however, is obtained, without greater labour of calculation, by considering these small arcs as so many parabolic arcs, according to the ingenious method invented by Simpson. We shall here collect the results, leaving it to the industry of the student, by a profitable exercise of calculation, to investigate the demonstration.

87. ***Proposition I.*** Let it be proposed to measure (Fig. 11.) the plane $XLL'X'$ .....

Let there be drawn any axis AS of the figure; let it be divided into an even number of equal parts AP, PQ, QR .....; and through the points of division let there be drawn the ordinates AL, PM, QN .... perpendicular to the axis. Let $AP = PQ = QR \ldots\ldots = h$; $AL = y_1, PM = y_2, QN = y_3 \ldots\ldots$ Then shall the area $ALXS$ be expressed by this formula,

$$\tfrac{1}{3}h\{y_1 + 4y_2 + 2y_3 + 4y_4 + 2y_5 + 4y_6 \ldots\ldots + y_n\}.$$

In like manner may be found the area of $AL'X'S$, and it may be added to that first found. If the lower curve $L'X'$ be equal to the upper LX, and be similarly situated with respect to the axis, it will be sufficient to double the value found for the area $ALXS$.

The above formula is deduced from the supposition, that the arc LMN is an arc of a parabola having MP for its diameter, and the arc NOX an arc of a parabola having OR for its diameter; and so on; a supposition which cannot lead to considerable error when the arcs are small.

88. *Coroll.* 1. In like manner for a solid, bounded by any curve surface, calling $s_1, s_2, s_3 \ldots.$ the areas of sections perpendicular to the axis AS, the solid content will be

$$\tfrac{1}{3}\,h.\,\{s_1 + 4s_2 + 2s_3 + 4s_4 + 2s_5 + 4s_6 \ldots. + s_n\}.$$

89. *Coroll.* 2. If the solid be produced by the revolution of the curve LX about the axis AS, we shall have $s = \pi y^2$, and the expression for the solid content becomes

$$\frac{\pi}{3}h.\{y_1^2 + 4y_2^2 + 2y_3^2 + 4y_4^2 + 2y_5^2 + 4y_6^2 \ldots. + y_n^2\}.$$

90. Scholium. If instead of considering the arcs LMN, $NOX \ldots.$ as parabolic arcs, the small arcs LM, MN, NO, $OX \ldots.$ had been taken as straight lines, we should have obtained for the value of the area $ALXS$ the following expression:

$$\tfrac{1}{2}.\,h\,\{y_1 + 2y_1 + 2y_2 + 2y_4 + 2y_5 + 2y_6 \ldots + y_n\};$$

and similar to this would have been the expression for the solid content.

91. *Proposition II.* Let $XLL'X'$ be a surface bounded by two equal and similar curves LX, $L'X'$, bisected by the axis AS: its centre of gravity will be in the axis AS, at a distance from the point A expressed by the following formula:

$$h.\,\frac{0.y_1 + 1.4y_2 + 2.2y_3 + 3.4y_4 + 4.2y_5 \ldots. + (n-1)\,y_n}{y_1 + 4y_2 + 2y_3 + 4y_4 + 2y_5 \ldots.. + y_n}.$$

92. *Coroll.* Similar to this is the expression for a solid symmetrical about AS, provided that the sections $s_1, s_2 \ldots$ perpendicular to the axis, be put for the ordinates y_1, y_2: and if the solid be generated by revolution, these sections will be represented by $\pi y^2, \pi y_2^2 \ldots$.

93. SCHOLIUM. If the curves LX were considered as polygons, with the small sides LM, $MN \ldots$ the formula would be this following,

$$\frac{1}{3}h.\frac{y_1 + 1.6y_2 + 2.6y_3 + 3.6y_4 + 4.6y_5 \ldots + (3n-4)y_n}{y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 \ldots + y_n}.$$

CHAP. XIII.

ON THE EQUILIBRIUM OF FORCES MEETING IN A POINT.

94. *Proposition I.* If a free point, or elementary particle of matter, be acted upon by any number of forces, and if the resultant of these forces be nothing, there be an equilibrium: and conversely.

95. *Coroll.* 1. All the forces having been reduced to three (80.) P, Q, R, parallel to three rectangular axes, in this case $\sqrt{P^2 + Q^2 + R^2} = 0$; whence $P = 0$, $Q = 0$, $R = 0$.

96. *Coroll.* 2. Let any number of forces, expressed by AB, AC, AD, &c. (Fig. 12.) be in equilibrium about the point A: then shall A be the centre of gravity of the points B, C, D, &c., considering these points as loaded with equal weights.

For let the system of points A, B, C, D, &c. be referred to three rectangular axes; let X, Y, Z, be the co-ordinates of the point A; x, y, z, those of the point B. The force AB being, by the hypothesis, expressed by AB, will be

$$= \sqrt{\{(x - X)^2 + (y - Y)^2 + (z - Z)^2\}};$$

and if it be resolved into three forces parallel to the three axes, these components will be $x - X$, $y - Y$, $z - Z$. In like manner the force AC may be resolved into three, $x' - X$, $y' - Y$, $z' - Z$; and the force AD into three $x'' - X$, $y'' - Y$, $z'' - Z$; and so on. Let n be the number of these forces AB, AC, AD, &c. which are in equilibrium. Wherefore (95.)

$$P = x + x' + x'' \ldots\ldots - nX = 0,$$
$$Q = y + y' + y'' \ldots\ldots - nY = 0,$$
$$R = z + z' + z'' \ldots\ldots - nZ = 0.$$

Therefore, the co-ordinates of the point A are these three,

$$X = \frac{x+x'+x''\ldots}{n};\quad Y = \frac{y+y'+y''\ldots}{n};\quad Z = \frac{z+z'+z''\ldots}{n};$$

which are (48.) precisely the co-ordinates of the centre of gravity of the points B, C, D, &c.

By reasoning conversely, it may be shewn that if the point A be the centre of gravity of the points B, C, D, &c., and if it be acted upon by the forces AB, AC, AD, &c., it will remain in equilibrium.

97. *Coroll.* 3. Hence the following theorems.

If a point be situated in the centre of gravity of a triangle, and be acted upon by three forces expressed by the straight lines drawn from it to the angles of the triangle, it will remain in equilibrium.

If, likewise, a point be situated in the centre of gravity of a triangular pyramid, and be acted upon by four forces expressed by the straight lines drawn from it to the angles of the pyramid, it will remain in equilibrium. For it is easy to demonstrate that the centre of gravity of a triangle, or of a triangular pyramid, is also the centre of gravity of equal weights placed at the angular points.

98. SCHOLIUM 1. If the point in which the forces meet, be stopped by a prop, which does not allow it any kind of motion, the equilibrium will subsist, however great be the number of acting forces; and the prop will be pushed, or drawn, with a force equal to the resultant of the forces applied.

99. SCHOLIUM 2. If the point be suspended from a fulcrum by means of a rod, it is requisite, for an equilibrium, that the resultant of the applied forces should pass through the fulcrum. And if the point hang by a thread, it is further necessary that this resultant do not urge the point towards the fulcrum, but in a contrary direction. The magnitude of the resultant is the magnitude of the force which stretches the thread or the rod, and also of the pressure which the fulcrum sustains.

100. *Proposition II.* If a point, or elementary particle of matter, lie upon a resisting surface, and if the resultant of the forces, applied to the point, be perpendicular to the surface, and press the point against it, there will be an equilibrium; and conversely.

In order to express, analytically, the condition of the resultant acting perpendicularly to the surface, it is necessary to premise the following geometrical lemma.

101. LEMMA. If a surface be determined by the equation $l\,dx + m\,dy + n\,dz = 0$; if k be a perpendicular to the surface at a point in it, the co-ordinates of which are x, y, z; if a, β, γ, be the angles which k makes with x, y, z, respectively; and if $l^2 + m^2 + n^2 = M^2$; then,

$$\cos. a = \frac{l}{M};\ \cos. \beta = \frac{m}{M};\ \cos. \gamma = \frac{n}{M}.$$

For let a, b, c, be the co-ordinates of the other extremity of k: wherefore,

$$k^2 = (x - a)^2 + (y - b)^2 + (z - c)^2;$$

$$\therefore\ k\,dk = (x - a)\,dx + (y - b)dy + (z - c)\,dz.$$

But $x - a = k \cos. a$; $y - b = k \cos. \beta$; $z - c = k \cos. \gamma$;

$$\therefore\ dk = dx \cos. a + dy \cos. \beta + dz \cos. \gamma.$$

But since k is perpendicular to the surface, it must be the greatest, or the least, of all straight lines drawn to the surface, from the point determined by the co-ordinates a, b, c. Wherefore $dk = dx \cos. a + dy \cos. \beta + dz \cos. y = 0$; from which equation, combined with the other, $l\,dx + m\,dy + n\,dz = 0$, eliminating dx, we have

$$(l \cos. \beta - m \cos. a)\,dy + (l \cos. \gamma - n \cos. a)\,dz = 0.$$

And this last equation must be true, whatever be the variations of dy, dz: wherefore,

$$l \cos. \beta - m \cos. a = 0,$$

and $l \cos. \gamma - n \cos. a = 0$:

also (27.) $\cos.^2 a + \cos.^2 \beta + \cos. \gamma^2 = 1$;

whence are deduced the three assigned values of cos. α, cos. β, cos. γ.

102. *Coroll.* Wherefore, if the resultant S of the forces applied to a material point, lying upon a surface, is to be perpendicular to the surface, resolving it into three forces P, Q, R, parallel to x, y, z, the following equations must obtain; viz.

$$P = S \cos.\alpha = \frac{lS}{M}; \quad Q = S \cos.\beta = \frac{mS}{M}; \quad R = S \cos.\gamma = \frac{nS}{M};$$

and, therefore, $P : Q : R :: l : m : n$.

It is clear that the resultant S is counteracted by the resistance of the surface, and represents the pressure which the surface itself sustains.

CHAP. XIV.

ON THE MOMENTUM OF ROTATION.

103. If a system, of invariable form, admit of rotatory motion about an axis, and if to any of its points a force be applied which acts in a plane perpendicular to the axis, the product of the force by the distance of its direction, from the axis of rotation, is called the *Momentum of the Force to turn the System about the Axis.*

Thus, let the force AP (Fig. 13.) be in a plane perpendicular to the axis Oo, and let AM be drawn in this plane perpendicular to AP; the product $AP.AM$ is the momentum of the force AP to turn the system about the axis Oo.

104. Scholium. If the force AP lie in a plane oblique to the axis Oo, it is necessary to resolve it into two; of which the first shall be parallel to Oo, and the second shall lie in a plane perpendicular to Oo. This second force is the only one which tends to turn the system about its axis; and its momentum is to be considered as the momentum of the force AP.

105. *Proposition.* If to a rigid system moveable about an axis, two forces be applied which tend to turn it in contrary directions, and if the momentums of the two forces be equal, there will be an equilibrium: and conversely.

Let the two forces AP, BQ (Fig. 13.) tend to turn the system about the axis Oo in contrary directions, with equal momentums $AP.AM$, $BQ.BN$. These forces act (104.) in planes perpendicular to the axis, and are therefore parallel to each other. Let any plane be supposed to be drawn through the axis, and let Ma, Nb, be the common sections of this plane with the two parallel planes, aMA, bNB, in which lie the forces AP, BQ. Take Ma, Nb equal, respectively, to MA, NB. At the extremity a, of Ma, suppose two opposite forces, ap, ap', equal to each other and to AP, to be applied in the plane aMA: and similarly to the extremity of Nb, let there be applied bq, bq', equal to each other, and to BQ, and acting in the plane bNB. Since these new forces mutually counteract each other, the resultant of the six forces AP, ap, ap', BQ, bq, bq', is the same as that of the two forces, AP, BQ.

Considering first, the two forces AP, ap', produce their directions until they meet in R; join R, M; and it is evident that RM bisects the angle PRp'. But the resultant of the two equal forces AP, ap', ought also (25.) to bisect their angle Prp'; wherefore, the resultant coincides with RM, and cuts the axis in the point M. It may similarly be shewn, that the resultant of the two BQ, bq', has the direction SN, and cuts the axis in the point N.

There remain, then, the two forces ap, bq, which by the construction, are parallel. Join a, b, and let ab cut the axis in G: and as, by the hypothesis, $AP . AM = BQ . BN$, therefore also $ap . aM = bq . bN$, and

$$ap : bq :: bN : aM :: bG : aG.$$

Wherefore (38.) the resultant of the forces ap, bq, cuts the axis in G.

Thus are the six forces, equivalent to AP, BQ, reduced to three, each of which cuts the axis, and therefore does not tend to turn the system. There will, therefore, be an equilibrium.

Again, if the momentums $AP . AM$, $BQ . BN$, be not equal, neither will $ap . aM$, $bq . bN$, be equal; and the resultant of the forces ap, bq, will not pass through G, but on the one side or the other of that point, in the straight line ab, so that the system will revolve. If, therefore, there be an equilibrium, the momentums of the forces applied must be equal to one another.

106. *Coroll.* 1. It is manifest, from the preceding demonstration, that as far as regards the equilibrium of rotation about the axis Oo, for the forces AP, BQ, we may substitute two others, as ap, bq, equal to the first, but with any different directions, provided that these directions lie always in planes perpendicular to the axis Oo, and preserve the same distances from the axis.

Thus may the above-mentioned forces be rendered all of them parallel to one another.

107. *Coroll.* 2. We may also, without disturbing the equilibrium of rotation, substitute for the force AP, another $a\pi$, of a different value, by changing the distance from the axis. It is only necessary that $AP.AM = a\pi.aM$; whence the same momentum is preserved.

108. *Coroll.* 3. If several forces act upon the system, tending to turn it in contrary directions, and if the sum of the momentums of those which tend to turn it in one direction, be equal to the sum of the momentums of those which tend to turn it in the contrary direction, there will be an equilibrium; and conversely.

For, let all the forces be reduced (106.) to forces parallel to one another; then, let all those, which tend to turn the system in the same direction, be reduced to their resultant P, applied at their centre, and let the momentum of this force be $P.m$. Similarly, let those which tend toward the contrary direction, be reduced to one Q, applied at their centre, and let its momentum be $Q.n$. Wherefore (55.) Pm is the sum of the momentums of the first, and Qn the sum of the momentums of the second: and, by the hypothesis, $Pm = Qn$; therefore (105.) there will be an equilibrium.

CHAP. XV.

PRINCIPAL PROPERTIES OF MOMENTUMS OF ROTATION.

109. **Lemma 1.** If through the point O (Fig. 14.) there be drawn three rectangular axes OX, OY, OZ, and if also, through the same point, there be drawn three other rectangular axes, OX', OY', OZ', each of which makes, with the three first, respectively, the angles α, β, γ; α', β', γ'; α'', β'', γ''; there will result these six equations,

$$\cos.^2\alpha + \cos.^2\beta + \cos.^2\gamma = 1.$$
$$\cos.^2\alpha' + \cos.^2\beta' + \cos.^2\gamma' = 1.$$
$$\cos.^2\alpha'' + \cos.^2\beta'' + \cos.^2\gamma'' = 1.$$
$$\cos.\,\alpha\,.\cos.\,\alpha' + \cos.\,\beta\,.\cos.\,\beta' + \cos.\,\gamma\,.\cos.\,\gamma' = 0.$$
$$\cos.\,\alpha\,.\cos.\,\alpha'' + \cos.\,\beta\,.\cos.\,\beta'' + \cos.\,\gamma\,.\cos.\,\gamma'' = 0.$$
$$\cos.\,\alpha'.\cos.\,\alpha'' + \cos.\,\beta'.\cos.\,\beta'' + \cos.\,\gamma'.\cos.\,\gamma'' = 0.$$

The three first result from Art. 28, the three last from the theorem proved in Art. 34, by which we have

$$\cos.\ X'OY' = \cos.\,\alpha.\cos.\,\alpha' + \cos.\beta.\cos.\,\beta' + \cos.\,\gamma.\cos.\,\gamma'.$$
$$\cos.\ X'OZ' = \cos.\,\alpha.\cos.\,\alpha'' + \cos.\beta.\cos.\,\beta'' + \cos.\,\gamma.\cos.\,\gamma''.$$
$$\cos.\ Y'OZ' = \cos.\,\alpha'.\cos.\,\alpha'' + \cos.\,\beta'.\cos.\,\beta'' + \cos.\,\gamma'.\cos.\,\gamma''.$$

But, by the hypothesis, the angles $X'OY'$, $X'OZ'$, $Y'OZ'$, are right angles, and their cosines are therefore equal to nothing. Therefore, &c.

110. *Coroll.* If we first consider the axes OX', OY', OZ', and then the others OX, OY, OZ, each of which makes, with the three first, the angles α, α', α''; β, β', β''; γ, γ', γ''; it is evident that, in addition to the six equations (109.), we shall also have these other six,

$$\cos.^2 \alpha + \cos.^2 \alpha' + \cos.^2 \alpha'' = 1.$$
$$\cos.^2 \beta + \cos.^2 \beta' + \cos.^2 \beta'' = 1.$$
$$\cos.^2 \gamma + \cos.^2 \gamma' + \cos.^2 \gamma'' = 1.$$
$$\cos. \alpha . \cos. \beta + \cos. \alpha' . \cos. \beta' + \cos. \alpha'' . \cos. \beta'' = 0.$$
$$\cos. \alpha . \cos. \gamma + \cos. \alpha' . \cos. \gamma'' + \cos. \alpha'' . \cos. \gamma'' = 0.$$
$$\cos. \beta . \cos. \gamma + \cos. \beta' . \cos. \gamma' + \cos. \beta'' . \cos. \gamma'' = 0.$$

111. LEMMA 2. Let the area a be in any plane drawn through the origin O; let this area be projected on the three planes YOZ, XOZ, XOY, into A, A', A'', and on the three planes $Y'OZ'$, $X'OZ'$, $X'OY'$, into B, B', B'': then

$$B = A \cos. \alpha + A' \cos. \beta + A'' \cos. \gamma.$$
$$B' = A \cos. \alpha' + A' \cos. \beta' + A'' \cos. \gamma'.$$
$$B'' = A \cos. \alpha'' + A' \cos. \beta'' + A'' \cos. \gamma''.$$

For, let OD (Fig. 14.) be perpendicular to the plane in which the area a lies. We shall have (32.)

$$\cos. X'OD = \cos. XOD \cos. \alpha + \cos. YOD \cos. \beta + \cos. ZOD \cos. \gamma .$$
$$\cos. Y'OD = \cos. XOD \cos. \alpha' + \cos. YOD \cos. \beta' + \cos. ZOD \cos. \gamma' .$$
$$\cos. Z'OD = \cos. XOD \cos. \alpha'' + \cos. YOD \cos. \beta'' + \cos. ZOD \cos. \gamma'' .$$

But it is demonstrated in Geometry, that the projection of a plane (a) upon any plane, is equal to the plane (a) multiplied by the cosine of the inclination of the two planes, or by the cosine of the angle contained by the perpendiculars to the two planes. Wherefore, since OD, perpendicular to the plane (a), makes with OX, perpendicular to the plane YOZ, the angle XOD, the projection of (a) on the plane $YOZ = a \cos. XOD$: and thus,

$$A = a \cos. XOD, \quad A' = a \cos. YOD, \quad A'' = a \cos. ZOD,$$
$$B = a \cos. X'OD, \quad B' = a \cos. Y'OD, \quad B'' = a \cos. Z'OD;$$

where, substituting for the cosines $X'OD$, &c. their values above determined, and for the products $a \cos. XOD$, &c. their values A, &c. there will result the values of the three projections B, B', B'', assigned in the Lemma.

112. *Coroll.* If instead of a single area a, there be several areas situated in different planes, all of them passing through O, and if, when each of these areas is projected on three rectangular planes, A, A', A'' denote the sums of the projections on the planes YOZ, XOZ, XOY, respectively, and B, B', B'', the sums of the projections on the planes $Y'OZ'$, $X'OZ'$, $X'OY'$, the equations of the preceding article will obtain with respect to the sums A, A', A'', B, B', B''.

For, from the form of each of these equations, it is plain, that, by writing down as many of them as there are areas a, and taking their sum, the resulting equation will preserve the same form; only, instead of the single projection of the area a, there will be introduced the sum of the projections of all the areas which accompany it.

113. Lemma 3. To the point A (Fig. 15.) determined by the co-ordinates x, y, z, let there be applied the force AB, which, resolved into three parallel to the axes OX, OY, OZ, gives the three forces P, Q, R, and straight lines OA, OB, having been drawn from the origin O, if the triangle OAB be projected on the three planes XOY, XOZ, YOZ, the three projections will be

$$\tfrac{1}{2}(Py - Qx),\ \tfrac{1}{2}(Rx - Pz),\ \tfrac{1}{2}(Qz - Ry).,$$

Let QOQ' be the projection of the triangle AOB, on the plane XOY. The ordinates QP, $Q'P'$, having been drawn, we shall have $OP = x$, $PQ = y$, $OP' = x + P$, $P'Q' = y + Q$. Now the triangle $QOQ' = POQ + PQQ'P' - P'OQ'$. But, $POQ = \tfrac{1}{2}xy$; $PQQ'P' = \tfrac{1}{2}PP'(PQ + P'Q') = \tfrac{1}{2}P(2y + Q)$; $P'OQ' = \tfrac{1}{2}(x + P).(y + Q)$: wherefore $QOQ' = \tfrac{1}{2}(Py - Qx)$. And in the same manner may the other two projections be found.

114. *Coroll.* 1. The momentums of the force AB, to turn the system about the axes OZ, OY, OX, are proportional to the projections of the area AOB, on the three planes XOY, XOZ, YOZ; each of these momentums being the double of the corresponding projection.

For the force AB, when it is resolved into the three P, Q, R, tends to make the system turn about the axis OZ, from Y towards X, with the momentum Py, and from A towards Y with the momentum Qx. Wherefore, the momentum of the force AB, to make the system turn about the axis OZ, from Y towards X, $= Py - Qx$. In the same manner, it may be shewn that the momentum of the force AB to turn the system about OY, from

X towards Z, $= Rx - Pz$; and that its momentum to turn the system about OX, from Z towards Y, $= Qz - Ry$. Therefore, its three momentums to turn the system about the axes OZ, OY, OX, in the direction XYZ, are $Py - Qx$, $Rx - Pz$, $Qz - Ry$. But (113.) the three projections of the area AOB, on the planes XOY, XOZ, YOZ, are $\frac{1}{2}(Py - Qx)$, $\frac{1}{2}(Rx - Pz)$, $\frac{1}{2}(Qz - Ry)$. Wherefore, &c.

115. *Coroll.* 2. In a system of any number of forces whatever, the sums of the momentums, with which these forces tend to turn the system about three rectangular axes OZ, OY, OX, are proportional to the sums of the projections of as many areas as there are forces, upon the planes XOY, XOZ, YOZ; each sum of momentums being the double of the sum of the corresponding projections.

116. *Proposition I.* If F, G, H, be the sums of the momentums with which the forces, applied to a system, tend to turn it about the axes OX, OY, OZ; and if F', G', H', be the sums of the momentums of the same forces, with respect to the axes (109.) OX', OY', OZ'; then

$$F' = F \cos. \alpha + G \cos. \beta + H \cos. \gamma,$$
$$G' = F \cos. \alpha' + G \cos. \beta' + H \cos. \gamma',$$
$$H' = F \cos. \alpha'' + G \cos. \beta'' + H \cos. \gamma''.$$

For, calling A, A', A'', the sums of the projections of all the areas, analogous to AOB, on the planes YOZ, XOZ, XOY; and B, B', B'', the sums of the projections of the same areas on the planes $Y'OZ'$, $X'OZ'$, $X'OY'$, we shall have (115.)

$$A = \tfrac{1}{2} F,\ A' = \tfrac{1}{2} G,\ A'' = \tfrac{1}{2} H;$$
$$B = \tfrac{1}{2} F',\ B' = \tfrac{1}{2} G',\ B'' = \tfrac{1}{2} H':$$

and, if these values of A, &c. B, &c. be substituted in the equations of Art. 111, there will result those expressed in the proposition.

117. *Coroll.* If the sum of the squares of these equations be taken, and reduced, by means of the equations in Arts. 109, 110, there results

$$F'^2 + G'^2 + H'^2 = F^2 + G^2 + H^2:$$

so that to whatever set of three rectangular axes, drawn through the same point, the momentums of the forces be referred, the sum of their squares is always the same.

118. *Proposition II.* The three momentums F, G, H, tending to turn the system about the three axes OX, OY, OZ, are equivalent to a single momentum $\sqrt{(F^2+G^2+H^2)}$ tending to turn it about a single axis, OX', which makes, with the three first, angles that have for cosines,

$$\frac{F}{\sqrt{(F^2+G^2+H^2)}},\quad \frac{G}{\sqrt{(F^2+G^2+H^2)}},\quad \frac{H}{\sqrt{(F^2+G^2+H^2)}}.$$

For, if $\cos.\ \alpha = \dfrac{F}{\sqrt{(F^2+G^2+H^2)}}$,

$$\cos.\ \beta = \frac{G}{\sqrt{(F^2+G^2+H^2)}},$$

$$\cos.\ \gamma = \frac{H}{\sqrt{(F^2+G^2+H^2)}},$$

and if, by the preceding formulas, the value of the momentums, F', G', H', be sought, with respect to the axis OX', and two other axes perpendicular to it, we shall have, first, (116.)

$$F' = \frac{F^2+G^2+H^2}{\sqrt{(F^2+G^2+H^2)}} = \sqrt{(F^2+G^2+H^2)};$$

and then (117.) $G'^2 + H'^2 = F^2 + G^2 + H^2 - F'^2 = 0$;

whence $G' = 0$, $H' = 0$. Wherefore, &c.

119. *Coroll.* 1. Conversely, if the forces applied to the system tend, with a momentum M, to make it revolve about an axis, which makes the angles α, β, γ, with three rectangular axes OX, OY, OZ; the momentum M will be equivalent to three momentums, $M\cos.\ \alpha$, $M\cos.\ \beta$, $M\cos.\ \gamma$, tending together to turn the system about the axes OX, OY, OZ, respectively.

120. *Coroll.* 2. Hence, it is evident that the different momentums which the forces, applied to a system, have to turn it at one time about different axes, may be compounded together, in the same manner as forces applied to a point.

Thus, if on the axes OX, OY, OZ, there be taken from the point O, three straight lines respectively equal to F, G, H, that

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is, to the momentums of the forces with respect to the several axes, and if, with these, as sides, a rectangular parallelepiped be constructed, its diagonal drawn from O will be the single axis, about which the system will tend to revolve with a momentum expressed by that same diagonal.

For, this diagonal will be $= \surd(F^2 + G^2 + H^2)$, and will make, with the three axes, OX, OY, OZ, angles which have for cosines

$$\frac{F}{\surd(F^2+G^2+H^2)}, \quad \frac{G}{\surd(F^2+G^2+H^2)}, \quad \frac{H}{\surd(F^2+G^2+H^2)}.$$

121. *Proposition III.* The momentums, F, G, H, being given, with which the forces applied to a system tend to turn it about the rectangular axes OX, OY, OZ, to find the momentums f, g, h, with which the same forces tend to turn it about three rectangular axes, parallel to the first, and drawn through a point, the co-ordinates of which are a, b, c.

In order to refer the momentums of the forces to these new axes, it is sufficient, in the values of the momentums F, G, H, instead of the co-ordinates x, y, z, to put $x-a$, $y-b$, $z-c$. And, since (114.) $F = \Sigma.Qz - \Sigma.Ry$, we shall have

$$\begin{aligned} f &= \Sigma.Q(z-c) - \Sigma.R(y-b) \\ &= F - c\,\Sigma.Q + b\,\Sigma.R: \\ g &= G - a\,\Sigma.R + c\,\Sigma.P, \text{ similarly,} \\ h &= H - b\,\Sigma.P + a\,\Sigma.Q, \end{aligned}$$

which was to be found.

122. *Coroll.* If $\Sigma.P = 0$, $\Sigma.Q = 0$, $\Sigma.R = 0$, and if also the momentums of rotation about the three rectangular axes be equal to nothing, so that $F=0$, $G=0$, $H=0$, the momentums of rotation about any other axes whatever will be equal to nothing.

CHAP. XVI.

ON THE EQUILIBRIUM OF A SYSTEM OF INVARIABLE FORM.

123. ***Proposition.*** If a rigid system be acted upon by several forces, each of which is decomposed into three P, Q, R, parallel to three rectangular axes, there will be an equilibrium, provided that, first, the sums of the forces parallel to each of the three axes are equal to nothing: secondly, the sums of the momentums of the forces to make the system turn about each of the axes are equal to nothing: and conversely.

For, the three forces applied to the system may be reduced to the three resultants $\Sigma . P$, $\Sigma . Q$, $\Sigma . R$, each of which tends to transfer the whole system, and every point of it, with a progressive motion parallel to its respective axis. If then each of these resultants be equal to nothing, no point of the system can (95.) have a progressive motion. Further, the momentums of the forces, to make the system turn about three rectangular axes, being equal to nothing, the momentums of the forces to make it turn about any other three such axes will also (122.) be equal to nothing; and therefore the system cannot have a rotatory motion. And it is plain that a rigid system cannot change its position, unless it be either by a progressive motion, or by a rotatory motion. Wherefore, &c.

Conversely, if there be an equilibrium, it is necessary that the system can neither receive a progressive, nor a rotatory motion. But it would receive a progressive motion, if any of the sums $\Sigma . P$, &c., were not equal to nothing; and it would receive a rotatory motion if any of the momentums referred to the three axes were not equal to nothing. Wherefore, &c.

124. *Coroll.* 1. Each of the forces having been resolved, as above, into three P, Q, R, and x, y, z, being the co-ordinates

of its point of application, the equilibrium of the system is determined by the six following equations,

$$\Sigma . P = 0; \ \Sigma . Q = 0; \ \Sigma . R = 0,$$
$$\Sigma . Py = \Sigma . Qx; \ \Sigma . Pz = \Sigma . Rx; \ \Sigma . Qz = \Sigma . Ry.$$

The first three indicate that the system cannot have a progressive motion, according to any one of the axes OX, OY, OZ. The three last indicate that the system cannot revolve about any one of the three axes OZ, OY, OX: and it has been shewn that, if these movements be prevented, every possible movement remains excluded.

125. *Coroll.* 2. If there be in the system a fulcrum, or fixt point, all progressive motion is prevented by the resistance of the fulcrum. So that it will then be sufficient that the forces P, Q, R, shall not cause any rotatory motion; and the equilibrium will be determined solely by the three equations,

$$\Sigma . Py = \Sigma . Qx; \ \Sigma . Pz = \Sigma . Rx; \ \Sigma . Qz = \Sigma . Ry.$$

126. *Coroll.* 3. If there be two fixt points, or, which amounts to the same, if there be a fixt axis, then the system can only have a rotatory motion about that axis: so that taking it for the axis of z, the single equation $\Sigma . Py = \Sigma . Qx$ will suffice for the equilibrium.

CHAP. XVII.

ON THE PRESSURES UPON THE SUPPORTS OF A RIGID SYSTEM IN A STATE OF EQUILIBRIUM.

127. *Proposition I.* If a rigid system, in a state of equilibrium, be sustained by a fulcrum, or fixt pivot, the same notation being used as before, the pressures p, q, r, which the fulcrum sustains in the direction of x, y, z, are determined by the equations

$$p = \Sigma . P ; \quad q = \Sigma . Q ; \quad r = \Sigma . R .$$

For the pressures sustained by the fulcrum are equal, and contrary, to those forces which must be applied there to maintain the equilibrium, if the fulcrum were removed. Now, by the three first equations of equilibrium (124.), it is manifest that the forces $\Sigma . P$, $\Sigma . Q$, $\Sigma . R$, ought to be there applied in directions contrary to x, y, z.

128. *Coroll.* The fulcrum sustains the stress of the powers P, Q, R, in the same manner as if they were immediately applied to it.

129. *Proposition II.* If a system be in equilibrium about the immoveable axes OZ (Fig. 15.), fixt in two points M, N; if p, q, r, be the pressures sustained in the directions of x, y, z, by the pivot at M, to which point belongs the abscissa $OM = \zeta$; and if p', q', r', be the pressures sustained similarly by the other pivot at N, to which belongs the abscissa $ON = \zeta'$, the values of these pressures are determined by the five following equations:

$$\begin{aligned} p + p' &= \Sigma . P. \\ q + q' &= \Sigma . Q. \\ r + r' &= \Sigma . R. \\ p\zeta + p'\zeta' &= \Sigma . Pz - \Sigma . Rx. \\ q\zeta + q'\zeta' &= \Sigma . Qz - \Sigma . Ry. \end{aligned}$$

For if the points M, N, be supposed to be removed, so that the system is less free, and if forces p, q, r, be applied, in directions contrary to the co-ordinates, at the point M, and forces p', q', r', at the point N, to maintain the equilibrium; when the values of these forces are deduced from the general equations of equilibrium (124.) there result the equations above written.

130. *Coroll.* 1. The third of the above equations shews that the whole axis OZ is pressed, in the direction of z, with a force $= \Sigma . R$. By the other four equations are easily determined the pressures sustained by each of the props in the directions of x and of y.

131. *Coroll.* 2. Hence may be easily solved the problem, in which are sought the pressures exerted by a door on two hinges that sustain it; or, more generally, the pressures exerted on the two pivots M, N of the axis, by a force parallel to the axis itself.

Let the force $= -R$; the distance of its direction from the axis $= x$; the distance between the two pivots $M, N = \zeta' - \zeta$.

Then, $r + r' = -R$; $p' = -p = \dfrac{Rx}{\zeta' - \zeta}$; $q' = q = 0$.

Thus the door presses, with its whole weight R, the vertical line of the hinges; and further, the upper hinge is drawn outwards, and the lower pushed inwards, each with a force $= \dfrac{RX}{\zeta' - \zeta}$.

CHAP. XVIII.

ON THE EQUILIBRIUM OF A RIGID SYSTEM ACTED UPON BY PARALLEL FORCES.

132. It is a property of a rigid system, acted upon by parallel forces, that (46.) all the forces may be reduced to one single resultant, applied at their centre, and equivalent to the sum of them all. Such is a heavy body; such is a mass, of which the particles tend all to move with equal velocities and in parallel lines. In these systems the determination of the conditions and of the properties of equilibrium is, for the most part, easy: as we shall see by going through some particular cases.

133. *Proposition I.* If a mass M, of which all the elements are impressed with a velocity V, be contiguous to a mass m, of which all the elements are impressed with a velocity u; and if the directions of the velocities V, u, be parallel to the straight line joining the centres of gravity of the two masses, and opposite to each other; there will be an equilibrium if the masses M, m, be reciprocally proportional to the velocities V, u; and conversely.

For all the forces, which act on the mass M, may be reduced to a single force applied at its centre of gravity and equal to $M.V$; likewise those which act on the mass m may be reduced to a single force at its centre of gravity and equal to $m.u$. These two single forces being opposite, there will be an equilibrium if $M.V = m.u$, or if $M : m :: u : V$; and conversely.

The product of the mass by the common velocity with which all its elements move, or tend to move, is called *quantity of motion*. This product, measuring the resultant of all the parallel forces which act on the elements of the mass, measures also the force of the mass itself.

134. *Proposition II.* A heavy body being suspended, or in any way sustained, on a fixt point, will remain in equilibrium, if the vertical drawn through the centre of gravity pass through that point: and conversely, (63. 99.).

135. *Proposition III.* A heavy body, placed upon a plane, will be in equilibrium if the plane be horizontal, and if further the vertical drawn through the centre of gravity fall within the space which is bounded by the straight lines that join the points of support: and conversely.

136. *Coroll.* 1. If the first condition be wanting, the body will slide along, grazing the plane: if the second be wanting, it will roll over that point of support, towards which the vertical, drawn from the centre of gravity, falls.

137. *Coroll.* 2. The heavy body presses the plane with all its weight; if the supports be two, or even three, provided that they do not lie in the same straight line, it is easy to determine (40. 42.) what part of the weight presses on each of the points of support; but in other cases (44.) the problem is indeterminate.

133. *Proposition IV.* If a heavy body be sustained on two inclined planes CA, CB (Fig. 16.) resting on the points of support A, B, there will be an equilibrium, if the vertical GV, drawn through the centre of gravity G of the body, lie between the two points A, B, and pass through the intersection P of the perpendiculars AP, BP, drawn from the points A, B, to the inclined planes: and conversely.

For the weight of the heavy body may be supposed to be applied to the point P of the vertical GV, and to be resolved into two forces, in the directions PA, PB, which acting perpendicularly against the planes, in the points of support, will (100.) be sustained.

And conversely, if there be an equilibrium, the perpendiculars AP, BP, must necessarily meet in a point of GV. For if we suppose the planes to be removed, and in their stead to be substituted at A, B, forces equal and contrary to the pressures which the heavy body exerts on the planes themselves, an equilibrium must subsist between the weight of the body, which acts according to the vertical GV, and these two forces, which act according to AP, BP. Now three forces, which are not parallel, cannot be in equilibrium, unless their directions meet in a point; because each of the forces must be equal and opposite to the diagonal of the parallelogram the sides of which represent the other two.

139. *Coroll.* 1. It is easy to determine the pressures on each of the points of support A, B.

Let P be the weight of the body, A, B, the pressures sought. Then, (17.)

$$P : A : B :: \sin. APB : \sin. BPG : \sin. APG.$$

Or, if the vertical CZ be drawn,

$$P : A : B :: \sin. ACB : \cos. BCZ : \cos. ACZ.$$

140. *Coroll.* 2. If the vertical GV do not pass through the intersection P of the perpendiculars AP, BP, but nearer to one of the points of support, for example, towards A, the body will slip between the two planes, the extremity at A descending, and the other extremity at B ascending.

141. *Coroll.* 3. If the body rest upon the planes not in one point only, but in more than one, or in an extended base, for example, in ab, $a\beta$, it will suffice for the equilibrium, that the vertical GV pass through the parallelogram $lmno$, formed by the perpendiculars drawn to the planes at the extremities of the bases ab, $a\beta$.

142. *Coroll.* 4. If one of the points of support were a fixt prop, which hindered the body from slipping along the plane, then it will suffice for the equilibrium, that the vertical GV pass between the points A, B.

In this case, also, the determination of the pressures is very easy: of which we shall give an example in the true solution of the following problem, which has been differently exhibited by different persons.

143. *Coroll.* 5. The beam AB (Fig. 17.) rests with its extremity A on the vertical plane ACX with its other extremity B on the horizontal plane CB, being propped at B by an obstacle, which hinders it from sliding along the plane: it is required to

find the pressures, or stresses, against the points of support A, B.

Through G, the beam's centre of gravity, draw the vertical GP meeting the horizontal line AP in P, and CB in T: and join PB. The weight of the body may be supposed to be applied at P, and to be resolved into two forces Aa, and Bb in the direction PB: and these will be the two pressures sought. If it be required to find, separately, the horizontal and vertical pressures exerted by the beam, resolve Bb into two BX, BZ, the former horizontal, the latter vertical. Suppose now the supports to be removed, and instead of them to be put the forces Aa, BX, BZ, acting in contrary directions; these forces must maintain an equilibrium with the weight P of the beam. And, since the system is all in one plane, we have from Art. 124, these three equations,

$$Aa = BX;\quad P = BZ;\quad P.CT + Aa.CA = BZ.CB.$$

$$\text{Whence}\quad Aa = BX = \frac{P.BT}{CA}.$$

Therefore, the prop B is pressed vertically by the whole weight of the beam; it, also, and the point A are pressed horizontally, in opposite directions, with a force expressed by

$$\frac{P.BT}{CA}.$$

Let $AB = a$, $AG = b$, the angle $BAC = \phi$. The horizontal pressure will be $P.\dfrac{a-b}{a}.\tan.\phi$.

And if the centre of gravity fall in the middle of the beam, the pressure becomes $\frac{1}{2}P\tan.\phi$.

CHAP. XIX.

ON SYSTEMS OF VARIABLE FORM.

144. Those conditions, which secure an equilibrium in rigid systems, are not sufficient to secure it in systems of a variable form. In these, besides a progressive and a rotatory motion, various other motions may take place, according to the particular form and disposition of the system; which requires other conditions of equilibrium, adapted to prevent all those partial motions of which each point is susceptible. Galileo's principle of virtual velocities is applicable also to systems of variable form, and supplies the equations of their equilibrium; but the general demonstration of this principle, and its application, often difficult, would draw us beyond the limits of an elementary treatise: besides that, for practical purposes, the consideration of some particular systems may suffice, in which the laws of equilibrium are easily deduced from the decomposition of forces. We shall, therefore, confine our attention to polygons, the angles of which may be increased, or diminished, by the action of forces applied to them; whether the sides be flexible or rigid; to which subject the following proposition will serve as an introduction.

145. *Proposition.* If two strings KA, $K'A$, meet in the point A (Fig. 18.) having their extremities fixt in the points K, K', and being drawn at A by the force P, the direction of which makes with the strings the angles a, a', and if T, T' be the tensions of the strings,

$$P : T : T' :: \sin. (a + a) : \sin. a' : \sin. a.$$

Let the force P, expressed by the straight line AP, be resolved into two, AT, AT', in the directions AK, AK' produced: it is manifest that the string AK is stretched with the force AT, and the other string AK' with the force AT': and (17.)

$AP : AT : AT' :: \sin. TAT' : \sin. PAT' : \sin. PAT$. Wherefore, &c.

146. *Coroll.* 1. The two strings necessarily lie in the same plane with the stretching force P.

147. *Coroll.* 2. If the angle of the two strings be very obtuse, their tensions are very great compared with the stretching force: yet that angle cannot vanish unless the force P become nothing. Hence, there is no tension sufficient to keep a heavy cord stretched in any straight line that is not vertical.

148. *Coroll.* 3. If the direction of the force P bisect the angle of the two strings, the strings are equally stretched, and the tension $= \frac{P}{2 . \cos. a}$.

149. *Coroll.* 4. If the knot A be a running noose, there cannot be an equilibrium, unless the direction of the force P bisect the angle of the strings.

For the force, acting on the knot A, will make it describe an ellipse having its focuses in K, K'. Now there cannot (100.) be an equilibrium, unless the direction of the force be perpendicular to this ellipse; in which case it bisects the angle made by the straight lines drawn to the two focuses.

CHAP. XX.

ON THE FUNICULAR POLYGON.

150. *Proposition.* Let $KABCDK'$ (Fig. 18.) be a polygon, the sides of which are a string, having its extremities fixt at k, k', stretched by the forces P, Q, R, S, which make, with the sides of the polygon, the angles α, α', β, β', γ, γ', δ, δ'; it is required to determine the conditions of equilibrium.

Resolve the force P, applied to the angle A, into two T, X, in the directions of the sides AK, AB, contiguous to A: in like manner, resolve the force Q into X', Y, in the directions of the sides BA, BC; and so on to the last force. Then (145.)

$$\begin{aligned}
P : T : X &:: \sin.\ (\alpha+\alpha') : \sin.\ \alpha' : \sin.\ \alpha,\\
Q : X' : Y &:: \sin.\ (\beta+\beta') : \sin.\ \beta' : \sin.\ \beta,\\
R : Y' : Z &:: \sin.\ (\gamma+\gamma') : \sin.\ \gamma' : \sin.\ \gamma,\\
S : Z' : T' &:: \sin.\ (\delta+\delta') : \sin.\ \delta' : \sin.\ \delta.
\end{aligned}$$

Now since the polygon is in equilibrium, the forces X, X', which draw AB in opposite directions, and each of which represents the tension of that side, must necessarily be equal; so must likewise Y, Y' be equal to one another, and Z, Z'. Deducing, therefore, from the preceding analogies, the values of these forces, we shall have

$$\frac{P \sin.\ \alpha}{\sin.\ (\alpha+\alpha')} = \frac{Q \sin.\ \beta'}{\sin.\ (\beta+\beta')},$$

$$\frac{Q \sin.\ \beta}{\sin.\ (\beta+\beta')} = \frac{R \sin.\ \gamma'}{\sin.\ (\gamma+\gamma')},$$

$$\frac{R \sin.\ \gamma}{\sin.\ (\gamma+\gamma')} = \frac{S.\ \sin.\ \delta'}{\sin.\ (\delta+\delta')}:$$

and these are the required conditions of equilibrium.

151. *Coroll.* 1. These formulas shew us at the same time the values of the tensions of the intermediate sides AB, BC, CD. The tensions also of the extreme sides AK, DK', are deduced from the same analogies,

$$T = \frac{P \sin. \alpha'}{\sin. (\alpha + \alpha')}; \quad T' = \frac{S \sin. \delta}{\sin. (\delta + \delta')}.$$

If the extremities K, K' of the polygon are not fixt, then, in order to maintain the equilibrium, there must be applied the forces $Kt = T$, $K't' = T'$, in the directions of the extreme sides AK, DK'.

152. *Coroll.* 2. If the forces P, Q, R, S, applied to the angles of the polygon, all of them acted upon one point, and if the forces Kt, $K't'$, which represent the tensions of the extreme sides of the polygon also acted upon that same point, all these forces would be in equilibrium about that point.

For resolving, as before, the forces P, Q, R, S, there would result so many pairs of forces equal and opposite to each other.

153. *Coroll.* 3. The polygon being in a state of equilibrium, and any angular point, as C, being assumed, the resultant of all the forces applied to the polygon, from its beginning K to the point C, is equal to the tension of the side CD, which terminates in C.

For, since the forces Kt and T, X and X', Y and Y', are equal and opposite to each other, the resultant of the forces Kt, T, X, X', Y, Y', Z, is evidently Z, which is the tension of the side CD.

154. *Coroll.* 4. If the directions of the forces P, Q, R, S, bisect the corresponding angles of the polygon, the whole polygon is equally stretched, with a tension expressed by any of the forces divided by the double of the cosine of the angle which it makes with the string where it is applied.

155. *Coroll.* 5. If the polygon be closed, and regular, the drawing forces are equal; each of them is to the tension of the polygon as the side of the polygon is to the radius of the circumscribed circle; and the sum of them all is to the tension of the polygon, as the perimeter of the polygon is to that radius.

156. SCHOLIUM. To the above-mentioned conditions of equilibrium we may add this other, that the forces, applied to the angles of the polygon, act from within towards without, tending to remove the angles themselves to a greater distance from each other: whilst the sides of the polygon are supposed to be formed of threads, or strings, incapable of being lengthened, but capable of bending, so that the extremities of the sides may approach each other.

But if the sides are so many inflexible bars, simply resting against each other, without joints at the angles, it will be necessary that the forces act from without towards within, urging the angles to approach nearer to each other. For the rigidity of the sides resists this approach, but does not hinder the system from separating at the angles.

Lastly, if the sides of the polygon are rigid bars, having joints at the angles, so that the angles may open or close, whilst the extremities of the sides cannot change their distance, it will be indifferent, as to the equilibrium, in what direction the forces act.

It is, however, manifest, that those conditions, which secure the equilibrium of a polygon of flexible sides, secure also that of a polygon of rigid sides, simply resting against each other; only that the directions of all the forces must be understood to be changed into their contraries.

CHAP. XXI.

ON THE FUNICULAR POLYGON LOADED WITH WEIGHTS.

157. ***Proposition.*** The forces P, Q, R, S, (Fig. 19.) applied to the polygon, being supposed to be parallel, to determine the conditions of equilibrium.

In this case, we shall have $\sin. \alpha' = \sin. \beta$; $\sin. \beta' = \sin. \gamma$; $\sin. \gamma' = \sin. \delta$. Multiplying by these, the equations (150.) of the polygon in its state of equilibrium, we shall have for the conditions of equilibrium, in a polygon drawn by parallel forces,

$$\frac{P \sin. \alpha \sin. \alpha'}{\sin. (\alpha + \alpha')} = \frac{Q \sin. \beta \sin. \beta'}{\sin. (\beta + \beta')}.$$

$$= \frac{R \sin. \gamma \sin. \gamma'}{\sin. (\gamma + \gamma')} = \frac{S \sin. \delta \sin. \delta'}{\sin. (\delta + \delta')};$$

or, in another form,

$$\frac{P}{\cot. \alpha + \cot. \alpha'} = \frac{Q}{\cot. \beta + \cot. \beta'}$$

$$= \frac{R}{\cot. \gamma + \cot. \gamma'} = \frac{S}{\cot. \delta + \cot. \delta'}.$$

Wherefore, each of the forces, P, Q, R, S, is proportional to the sum of the co-tangents of the angles into which its direction divides the angle of the polygon.

158. *Coroll.* 1. In this case, the directions of the forces, and the whole polygon, necessarily lie in the same plane.

For the second force BQ, which, by the hypothesis, is parallel to AP, must (146.) be in the same plane in which are the two sides AK, AB, and the force AP. And in this plane will also lie the third side BC; and therefore also, the third force

CR; and so on to the last of them. The plane will be vertical, if the polygon be loaded with weights, as we here particularly suppose.

159. *Coroll.* 2. Resolve the tension Kt, of the first side, into two forces, the one, A, horizontal, the other, V, vertical. In like manner, resolve the tension $K't'$ into two, A', horizontal, V', vertical. And since (152.) all the forces acting on the polygon, if they be supposed to be transferred to any one point of the system, must be in equilibrium, it is plain that $A = A'$; $V + V' = P + Q + R + S$.

$$\text{Also } A = A' = T \sin. \alpha = (151.) \frac{P . \sin. \alpha \sin. \alpha'}{\sin. (\alpha + \alpha')} = \frac{P}{\cot. \alpha + \cot. \alpha'},$$

160. *Coroll.* 3. In general, the values of the tensions of the sides KA, AB, BC, CD, DK', expressed (150.) by T, X, Y, Z, T', may be deduced as is shewn above in Art. 151. Resolving each of these tensions into two forces, the one horizontal, the other vertical, the horizontal forces are found to be $T . \sin. \alpha$, $X . \sin. \beta$, $Y . \sin. \gamma$, $Z . \sin. \delta$, $T' . \sin. \delta'$; which expressions, by substituting in them the values of T, X, &c. become

$$\frac{P}{\cot. \alpha + \cot. \alpha'}, \ \frac{Q}{\cot. \beta + \cot. \beta'}, \ \&c.;$$

and, therefore, (157.) they are all equal to one another, and to A.

Hence, the horizontal tension is constant through the whole polygon, and equal to one of the weights, as P, divided by the sum of the co-tangents of the respective angles α, α'.

161. *Coroll.* 4. The absolute tensions of the strings KA, AB, &c. are, in their order, $\frac{A}{\sin. \alpha}$, $\frac{A}{\sin. \beta}$, $\frac{A}{\sin. \gamma}$, $\frac{A}{\sin. \delta}$, $\frac{A}{\sin. \delta'}$.

162. A remarkable property of this polygon is, that the directions of the extreme sides KA, $K'D$, produced, meet in some point of the vertical drawn through the common centre of gravity of the weights P, Q, R, S.

This may be demonstrated by calculation, but more readily by considering, that this system is reducible to three forces: one of them is the resultant, or sum, of the weights P, Q, R, S,

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which acts according to the vertical drawn through the centre of gravity: the other two are the tensions *Kt*, *K′t′*, of the strings attached to the fixt points, and these act in the directions *AK*, *DK′*. Now, for three forces to be in equilibrium, it is necessary that their directions meet in a point.

163. Scholium 1. If the sides of the polygon are not flexible strings, but bars, or rods, resting against each other, the forces ought (156.) to act in contrary directions, with respect to the angles of the polygon. Hence, if the polygon is loaded with weights, it will be necessary that it be inverted, so that the angle, which each side makes with the direction of the weight, may be the supplement of that which it before made: and instead of hanging upon the props *K*, *K′*, it must now be erected upon them.

In other respects, the conditions and the properties of equilibrium are the same as before, and by the same formulas, the tensions of the sides are determined, which, in this case, act in a contrary direction, and ought rather to be called pressures or stresses.

164. Scholium 2. The rigid polygon, besides being loaded at its angles, may be also loaded in one or more points of its sides. Then, the weights, which load each side, must be resolved (40.) into two applied at the extremities of that side. By this method, all the weights which load the polygon, may always be understood to be applied at its angles.

CHAP. XXII.

ON THE FUNICULAR CURVE.

165. *Proposition I.* To find the curve of equilibrium of a flexible thread AMB (Fig. 20.) acted upon, in all its points, by given forces.

Let $AP = x$, $PM = y$, $AM = s$; and taking for the element of the thread the small arc $Mm = ds$, let it be considered as constant. Let P, Q, be the forces applied to the point M, parallel to x and y respectively; and then (133.) the forces acting on the element Mm, will be Pds, Qds. Let T be the tension of the element, or indefinitely small side, Mm; which tension is exerted in the direction of that side, or according to the tangent Mt of the curve. Considering the curve as a funicular polygon of an infinite number of sides, the tension T must (153.) be the resultant of all the forces applied to the curve from the point A to the point M. And if the tension T be resolved into two forces, the one in the direction of x, the other in that of y, the former will be $-\frac{Tdx}{ds}$, the latter $-\frac{Tdy}{ds}$. Wherefore,

$$-\frac{Tdx}{ds} = \int Pds; \quad -\frac{Tdy}{ds} = \int Qds:$$

from which two equations if T be eliminated, there will result the required equation of the thread

$$dy\int Pds - dx\int Qds = 0.$$

166. *Coroll.* 1. From the same equations may be also found the value of T: for squaring them, and taking the sum,

$$T^2 = (\int Pds)^2 + (\int Qds)^2.$$

Another expression for T may also be found by differentiating these two equations and taking their sum, after having multiplied the first by dx, and the second by dy. For, since

$$dx^2 + dy^2 = ds^2,$$

$$\therefore\ dx\,ddx + dy\,ddy = ds\,dds = 0;$$

$$\therefore\ -dT = Pdx + Qdy;$$

$$\therefore\ T = C - \int(Pdx + Qdy).$$

167. *Coroll.* 2. If the thread be fixt to a nail at the extremity A, it is manifest, that amongst the forces applied to the curve from A to M, is included also the force At', which must be substituted in A, to maintain the equilibrium, if the point A be free; and that this force At', is equal and contrary to the resultant of all the forces applied to the thread from A to B. Therefore this resultant will pass through the point of suspension A, and will there touch the curve AMB.

168. *Coroll.* 3. The problem will scarcely be more difficult, if the forces, which bend the thread, act in different planes. In that case, $-dT = Pdx + Qdy + Rdz$; and the double curvature of the thread will be expressed by any two of the three following equations:

$$dy\int Pds - dx\int Qds = 0,$$

$$dz\int Pds - dx\int Rds = 0,$$

$$dz\int Qds - dy\int Rds = 0.$$

169. *Coroll.* 4. The curve, expressed by the equation of Art. 165, will also (156.) be the curve of equilibrium of an arch, composed of rigid and very small sides resting against each other. It is only requisite that the forces act in the contrary direction, that is, from without towards within.

170. *Coroll.* 5. If a flexible sheet be fastened to the circumference of a circle, of which CB (Fig. 20.) is the radius, and if, in every point of the circumference in it, which has $PM = y$ for radius, it be acted upon by the forces P, Q: it is required to find the curve AMB, by the revolution of which about AC, is generated the surface of the sheet in its state of equilibrium.

Considering the equilibrium of the ungula BAb, the trapezium Mq may be taken as its element. And since $Mm = ds$, and Mp is proportional to y, this trapezium may be expressed by $kyds$, k being a constant coefficient. Hence, the forces acting on Mq will be $Pkyds$, $Qkyds$; and the tension, $T.Mp$, may be expressed by the product Tky. Making these changes in the equations of equilibrium, we shall have $-(Tdy) = Pydx + Qydy$; and there will result for the equation of the curve AMB,

$$dy \int Pyds - dx \int Qyds = 0.$$

If the forces P, Q, be turned inwards, the same curve AMB will produce, by its revolution, a vaulted shell, such that, being composed of an infinite number of small and rigid faces, as Mq, simply resting against each other, these by their mutual action will sustain one another.

171. SCHOLIUM. If the sheet entirely surrounds the circumference of the circle, so that, when it is cut by a plane perpendicular to the axis AC, the section is a complete circle, having y for its radius, there may then be an equilibrium, even when $dy \int Pyds - dx \int Qyds < 0$. For, supposing the sheet to be in a state of equilibrium, and to have for its equation

$$dy \int Pyds - dx \int Qyds = 0,$$

imagine any force to act on the element Mq, in the direction of PM, and an equal force to be applied, all round, to the other elements, which, together with Mq, compose the circular zone having y for its radius. It is manifest, that the equilibrium will not be disturbed; since it is supposed that the fibres of the sheet do not admit of being lengthened in any direction, and therefore the zone does not admit of being enlarged. The effect of this additional force is confined to the producing of a tension in the perimeter of the zone, which is measured in the manner laid down in Art. 155; but the equilibrium is completely preserved. Wherefore, in the equation $dy \int Py\,ds - dx \int Qyds = 0$, the value of $\int Qyds$ may be increased in any manner, and to any extent whatever, without destroying the equilibrium. It is possible, therefore, that $dy \int Pyds - dx \int Qyds < 0$.

This will not be so, if there be applied to the zone passing through Mq, a force in the direction of MP; because there is nothing to hinder the whole zone from being reduced to a straighter circle by collapsing. Hence it is never possible that

$$dy \int Pyds - dx \int Qyds > 0.$$

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By applying a similar mode of reasoning to the vaulted shell, composed of solid and incompressible faces, and closed in all around, it may be shewn, that by adding any force whatever, in the direction of MP, to all the points of each zone, the equilibrium is not disturbed. In this case, therefore, the value of $\int Qyds$ may be diminished at pleasure, and it is possible that $dy\int Pyds - dx\int Qyds > 0$.

Upon the whole, therefore, for the entire sheet in a state of equilibrium, the equation of the curve AMB, proper for the generation of its surface, is

$$dy\int Pyds - dx\int Qyds = 0, \text{ or } < 0.$$

And for the whole vaulted shell, it is

$$dy\int Pyds - dx\int Qyds = 0, \text{ or } > 0.$$

172. *Proposition II.* To find the curve of equilibrium of a thread, every where acted upon by forces perpendicular to the thread.

This problem is a particular case of the preceding. Let Nds be the force perpendicular to the curve which urges the element ds. Resolving it into two, Pds, Qds, parallel to x and y, we shall have (35.) $Pds = Ndy$, $Qds = -Ndx$. Therefore (165.) the equation of the curve will be

$$dy\int Ndy - dx\int Ndx = 0.$$

173. *Coroll.* 1. Also (166.) $-dT = Pdx + Qdy = 0$; whence T is a constant quantity; and, therefore, in a thread stretched by normal forces, the tension is every where the same.

174. *Coroll.* 2. Let ϕ be the angle rMm, which the element of the curve ds makes with the direction of x; then $dx = ds$ cos. ϕ, $dy = ds$ sin. ϕ. But (165.) $-\frac{Tdx}{ds} = \int Pds$; or $-T\cos.\phi = \int Ndy$; and, differentiating,

$$Ndy = Td\phi \sin.\phi = \frac{Td\phi\, dy}{ds};$$

whence there results the equation, of a very simple form,

$$Nds = Td\phi.$$

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175. *Coroll.* 3. Let R be the radius of curvature of the curve, at the point M; $R = \frac{ds}{d\phi}$: whence the equation (174.) gives us $N = \frac{T}{R}$; and hence, we deduce this most remarkable property, that the force applied to the curve is every where reciprocally proportional to the radius of curvature: and conversely.

176. *Coroll.* 4. If a string, applied to the periphery of a resisting curve, is drawn at its two extremities by two forces, in the state of equilibrium, the two forces must be equal; the string is every where equally stretched; and the pressure which it exerts on each point of the curve, is inversely proportional to the radius of curvature.

For the pressure of the string on the curve, to which it clings, is (100.) every where perpendicular to the curve itself. Now the equilibrium will subsist, if we suppose, that the curve being removed, there are applied, to all the points of the string, forces equal and contrary to the pressures which the string exerted on the points of the curve immediately below it. The case is, therefore, reduced to that of a string stretched by forces perpendicular to it. Wherefore, &c.

CHAP. XXIII.

ON THE CATENARY.

177. *Proposition I.* To find the curvature of a heavy cord, or chain, KAK' (Fig. 21.) suspended from two fixt points K; K'.

This is only a particular case of the preceding problem (165.). In order to obtain the equation of the curve in its simplest form, assume, for the origin of the abscissas, the lowest point A of the chain, and for the axis, the vertical straight line AB, put $AP = x$, $PM = y$, $AM = s$. The equilibrium of the chain will subsist, if the whole arc AK', be supposed to be cut off and removed, provided that there is applied to the point A, a force At, equal to the horizontal tension of the curve, which we know (160.) to be constant, and which we have called A. Now if the weight of the arc AM be called Π, we shall have $\int P ds = -\Pi$, $\int Q ds = -A$. Whence (165.) the equation of the curve will be $-\Pi dy + A dx = 0$; or, $\Pi dy = A dx$.

178. *Coroll.* 1. If the lowest point A is loaded with a particular weight V, then $\int P ds = -\Pi - V$, and the equation of the catenary is

$$\Pi dy + V dy = A dx.$$

179. *Coroll.* 2. The tension at any point M will be (166.) $T = \sqrt{(\Pi^2 + A^2)}$; whence it goes on always increasing from A to K.

180. *Coroll.* 3. If the weight of the chain be uniform, Π will be proportional to s, and may be made $= ks$, k being a constant factor. Hence the equation of the homogeneous catenary will be $ksdy = Adx$, instead of which we may write simply $sdy = Adx$; recollecting that the constant quantity A then denotes the horizontal tension of the chain divided by the coefficient k.

181. *Coroll.* 4. In order to bring this equation into finite terms, it is to be observed that, since

$$ds = dy\sqrt{\left(\frac{dx^2}{dy^2}+1\right)},$$

if we put for $\frac{dx}{dy}$ its value $\frac{s}{A}$, we shall have, between y and s, the equation

$$dy = \frac{A\,ds}{\sqrt{(A^2+s^2)}},$$

which being integrated so that $y = 0$, may give $s = 0$, we have

$$y = A\log.\frac{s+\sqrt{(A^2+s^2)}}{A}.$$

Hence, we have between x and s, the equation

$$dx = \frac{s\,ds}{\sqrt{(A^2+s^2)}},$$

which being integrated so that $x = 0$, may give $s = 0$, we have $A + x = \sqrt{(A^2+s^2)}$; whence $s = \sqrt{(2Ax+x^2)}$.

Hence, lastly, the equation between x and y is

$$y = A\log.\frac{A+x+\sqrt{(2Ax+x^2)}}{A}.$$

182. *Coroll.* 5. The length, therefore, of the homogeneous catenary, may always be found; the arc s being a mean proportional between x and $2A+x$.

183. *Coroll.* 6. If it be required to describe a catenary, by determining any number of its points, it is necessary that the position of the extremities K, K', and the length KAK' of the chain be given. Let then $KQ=a$, $QK'=b$, $KAK'=l$, be given; and let $KB=m$, $BA=n$, $KMA=h$, be the quantities to be determined. If, in the equations

$$s = \sqrt{(2Ax+x^2)},\ y = A\log.\frac{A+x+s}{A},\quad x=n,$$

then $y=m$, and $s=h$; and if $x=n-b$, then $y=a-m$, $s=l-h$. Hence, we have four equations, by which are determined the quantities m, n, h, A.

Thus the position of the axis AB having been found, and the constant quantity A having been determined, all the points of the catenary will be determined by means of the equation between x and y.

184. *Coroll.* 4. Further, the mechanical description of the homogeneous catenary, is rendered very easy, by marking, on a vertical plane, the track of an uniform chain attached to two fixt points. When the sagitta is sufficiently small, the course of this curve approaches nearly to that of the cycloid, or to that of the parabola. For in that case, the arc s differs little from the ordinate y; and the equation, $s dy = A dx$, may be exchanged for $s ds = A dx$, which belongs to the cycloid; or for $y dy = A dx$, which belongs to the parabola.

Galileo is, without foundation, repeatedly accused of having confounded the catenary with the parabola; he has merely noticed the similarity which exists between these two curves.

185. *Coroll.* 8. We may determine, very nearly, by a very easy rule, the altitude of the bend, which a cord, attached to two points in the same horizontal straight line, will make in its bisection, when it is stretched by a weight very heavy in comparison of its own weight.

Let L be the length of the cord, p its weight, and P the stretching weight; we shall have, very nearly, $s = \frac{1}{2} L$, and $\frac{s}{A} = \frac{p}{2P}$. And the equation used (184.) for the purpose of approximation, $s ds = A dx$, will give the sagitta $x = \frac{s^2}{2A} = \frac{pL}{8P}$.

186. *Proposition II.* If from the circumference of an horizontal circle, having BK (Fig. 21.) for its radius, a sheet be suspended, which is loaded with weights uniformly distributed in every horizontal section, it is required to find the curve AMK, which, by its revolution about the axis AB, would generate the surface of the sheet.

The very small ungula KAk, may be considered as a catenary fixt at the two extremities Kk and A. Let the trapezium Mq be taken for the element of this ungula. And, because its base Mp is proportional to y, the weight of the trapezium Mq may be expressed by $y d\Pi$; whence the weight of the partial ungula MAP will be $\int y d\Pi$. If, then, in the equation of the

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linear catenary (177.) instead of Π there be put $\int y\,d\Pi$, we shall have the equation sought of the curve AMK; which will be

$$dy\int y\,d\Pi = A\,dx.$$

187. *Coroll.* 1. If the lowest point A were loaded with a particular weight V, the equation would be

$$dy\int y\,d\pi + V\,dy = A\,dx.$$

188. *Coroll.* 2. If the weight be uniformly diffused throughout the whole surface of the sheet, $d\Pi$ will be proportional to ds, and we shall have $k\,dy\int y\,ds = A\,dx$; or, more simply (180.) $dy\int y\,ds = A\,dx$.

189. SCHOLIUM 1. If Fig. 21. be inverted, so that the angle, which each element of the curve makes with the direction of the weight, may be the supplement of that which it made before, it is manifest (163.) that the curve of equilibrium of a vaulted surface, composed of small rigid faces, heavy or loaded, and resting against each other is a catenary. And, calling Π the weight of the arc, its equation is $\Pi\,dy = A\,dx$.

It is also manifest, that the surface of equilibrium of a cupola, composed of very narrow, heavy, or loaded circular zones, is generated by the revolution of a curve, which, calling $y\,d\Pi$ the weight of an elementary zone, has for its equation $dy\int y\,d\Pi = A\,dx$.

190. SCHOLIUM 2. Conformably to what was said in Art. 171, it will be proper, in the last place, to remark, that if the sheet extend round the whole circumference of the circle, and be composed of fibres, which do not admit of being lengthened, by stretching, in any direction, it is also possible that

$$dy\int y\,d\Pi < A\,dx:$$

and, on the other hand, if the cupola be entire, it is possible that

$$dy\int y\,d\Pi > A\,dx.$$

Hence, the equation, or rather the general condition of the curves, which, by revolving about a vertical axis, can generate the equipoised surface of a heavy sheet, is

$$dy\int y\,d\Pi = A\,dx, \text{ or } < A\,dx;$$

and the general condition of the curves, which can generate the surface of an equipoised cupola, is

$$dy\int y\,\Pi = A\,dx, \text{ or } > A\,dx.$$

CHAP. XXIV.

ON THE ELASTIC CURVE.

191. If a straight rod, or lamina AL (Fig. 22.) when bent into the form AB, exerts a force to recover its former straight position, the rod is called *Elastic*, and the force, which tends to straighten it, is termed *Elasticity*.

This force, acting upon every small arc Mm of the lamina, tends to make the arc turn about its extremity M, so as to bring it into the direction of the contiguous small arc μM. Hence, it acts with a certain momentum of rotation, with respect to the point M, or rather with respect to an axis perpendicular to the plane, in which the lamina lies, and passing through M. As to the value of this momentum, we shall assume for the present, the following hypothesis.

192. *Hypothesis.* The momentum of elasticity for every point M, of the bent elastic lamina AMB, is inversely proportional to the radius of curvature of the curve AMB at the point M.

Let R be this radius. The momentum of elasticity will be expressed by $\frac{E}{R}$; E being a quantity which is constant for the same lamina when it is homogeneous, and of uniform thickness, but which may be different in different laminas.

193. ***Proposition I.*** To find the curve of equilibrium of an homogeneous elastic lamina, acted upon in all its points by given forces.

Since the small arc Mm (Fig. 22.) tends to turn about the point M, and the momentum of this force is $\frac{E}{R}$, in order to the equilibrium of the curve it is requisite, that the resultant of all the forces, applied to it from the point A to the point M, produce not only the tension T, according to the tangent Mt, but also a normal force N, which tends to turn the arc about M in a contrary direction, and with an equal momentum. Let then $N.\omega$ be the momentum of this force, ω being the distance of its direction from the point M. Wherefore, $N\omega = \frac{E}{R}$. And because, in passing from the point M to the indefinitely near point m, the momentum of elasticity becomes $\frac{E}{R} + d.\frac{E}{R}$, and the momentum of the force N becomes $N(\omega + ds)$, for the point m we shall necessarily have $N(\omega + ds) = \frac{E}{R} + d.\frac{E}{R}$; whence $Nds = d.\frac{E}{R}$.

Now, preserving the notation of Art. 165, the resultant of all the forces applied to the lamina from A to M, is reduced to two forces $\int Pds$, $\int Qds$, parallel respectively to x and y. For these forces if other two be substituted, the one T, according to the tangent, the other N, according to the normal, there results (34.)

$$T = \frac{dx\int Pds + dy\int Qds}{ds}; \quad N = \frac{dy\int Pds - dx\int Qds}{ds}.$$

The first formula gives us the tension of the lamina in M: the second gives us the normal force N, which, substituted in the equation $Nds = d.\frac{E}{R}$ will give for the equation of the curve AMB

$$dy\int Pds - dx\int Qds = d.\frac{E}{R}.$$

194. Scholium. If the lamina were not homogeneous, or if the momentum of elasticity were not supposed to be inversely proportional to the radius of curvature, then, instead of expressing that momentum by $\frac{E}{R}$, we might denote it simply by E; under-

standing by E, no longer a constant but a variable quantity, which is a function of s: and, instead of the equation of Art. 193, we should have

$$dy\int P\,ds - dx\int Q\,ds = dE.$$

195. *Proposition II.* To find the figure of an horizontal elastic lamina AL, (Fig. 22.) fixt at its extremity A, after it has been bent into the figure AMB by a force BS, applied to the other extremity B.

The lamina being fixt in A, we may suppose (167.) such a force applied to the point A, as would maintain the equilibrium, if the point became free. If the direction BS pass through A, the force to be applied at A will be equal and contrary to BS. But if not, the lamina must needs be fixt, not in the single extreme point A, but in the extreme very small line Aa, or in the two points A, a. Then, resolving (40.) the force BS into two ah, AH, parallel to it, and applied at a, A, the forces ah, AH, will be capable of maintaining the equilibrium, the system being rendered free. But (37.) $ah - AH = BS$. Wherefore, the force, which must be supposed to be applied to the small line Aa, will be equal and parallel to BS, but will have a contrary direction.

Resolve now BS into two forces, the one, $-F$, horizontal, the other, G, vertical. These two forces will act at the very beginning Aa of the curve; so that $\int P\,ds = -G$, $\int Q\,ds = F$; and (193.) the equation of the curve AMB will be

$$d.\frac{E}{R} = -F\,dx - G\,dy;\text{ whence }\frac{E}{R} = C - Fx - Gy.$$

196. *Coroll.* 1. To determine the constant quantity C, it must be observed that, in the last element CB, the inflexion must vanish, because the force B, immediately applied to it, has but an indefinitely small momentum to make it turn about c. Wherefore, calling $AC = a$, $CB = b$, when $x = a$, and $y = b$, then $\frac{E}{R} = 0$. The constant quantity having been thus determined, the equation will be

$$\frac{E}{R} = F(a - x) + G(b - y).$$

197. *Coroll.* 2. If the lamina be bent by the force of a weight G, hanging from the extremity B, then $\frac{E}{R} = G(b-y)$; whence the radius of curvature, at any point M of the lamina, is inversely proportional to the horizontal distance of that point from the extremity B.

198. *Coroll.* 3. If the element ds be assumed to be constant, we have $R = \frac{dsdy}{ddx}$; whence, the equation between the co-ordinates of the curve is

$$Eddx = G(b-y)\,dsdy,$$

which is not integrable in finite terms.

But, if the flexure of the lamina be very small, we may make $ds = dy$; and then, since the element dy may also be considered as constant, we shall have

$$E\,.\,d\,.\frac{dx}{dy} = G(b-y)\,dy;$$

and, integrating so that $y = 0$ may render $\frac{dx}{dy} = 0$, there results

$$\frac{Edx}{dy} = Gby - \tfrac{1}{2}Gy^2;$$

again, integrating,

$$6\,Ex = 3\,Gby^2 - Gy^3,$$

an equation belonging to a cubical parabola.

199. *Proposition III.* To find the figure of the elastic lamina AB (Fig. 23.) placed vertically, and bent by the force of a weight G, placed on its top.

Here it is manifest, that, if $AP = x$, $PM = y$, $AM = s$, then $\int Pds = G$, $\int Qds = 0$; whence (193.) $d\,.\frac{E}{R} = Gdy$; and, integrating, $\frac{E}{R} = Gy$. Nor is any correction necessary; because, for the reason before assigned (196.) at the point A where $y = 0$, there $\frac{E}{R} = 0$. Wherefore, at any point M of the bent lamina, the radius of curvature is inversely proportional to the depth PM of the bend.

200. *Coroll.* 1. Making ds constant, we have $R = \frac{dsdx}{ddy}$; whence the equation between the co-ordinates of the curve will be $Eddy = -Gydsdx$. If the flexure be very small, we may assume the element $dx = ds$, and constant; and the equation will become

$$Gy + \frac{Eddy}{dx^2} = 0.$$

Let $RS = f$, the greatest depth of the curve, so that the tangent at S is vertical. Multiplying the equation by $2dy$, and integrating it so that $y = f$ may render $\frac{dy}{dx} = 0$, we shall have $\frac{Edy^2}{dx^2} = G(f^2 - y^2)$; or $dx = \sqrt{\frac{E}{G}} \cdot \frac{dy}{\sqrt{(f^2 - y^2)}}$: and integrating again, $x = \sqrt{\frac{E}{G}}$. Arc. sin. $\frac{y}{f}$; or $y = f$ sin. $x\sqrt{\frac{E}{G}}$.

201. *Coroll.* 2. If $x = AB = a$, then must $y = 0$. Wherefore f sin. $a\sqrt{\frac{G}{E}} = 0$; whence $a\sqrt{\frac{G}{E}} =$ arc . sin. $0 = \pi$; and $G = \frac{E\pi^2}{a^2}$. This, then, is the value which the weight G must have, in order to produce in the lamina a very small flexure.

If $G < \frac{E\pi^2}{a^2}$, it cannot satisfy the equation f. sin. $a\sqrt{\frac{G}{E}} = 0$, unless $f = 0$. Thus $y = 0$; so that the lamina will not bend at all any where.

And if $G > \frac{E\pi^2}{a^2}$, the equation, $f \,.\, \text{sin.}\, a\sqrt{\frac{G}{E}} = 0$, will lead to the same conclusion, and will shew that the lamina cannot bend. But such a conclusion cannot, in this case, be at all admitted; it being manifest that the lamina, once bent, must, by the addition of the new weight, bend the more. Here we ought rather to say, that, the flexure being no longer very small, the equation, $y = f\, \text{sin.}\, x\sqrt{\frac{G}{E}}$, is no longer proper to represent the figure of the arc of the lamina.

202. *Coroll.* 3. If $G = \frac{E\pi^2}{a^2}$, the ordinate $y = f$, answers to the abscissa $x = \frac{1}{2}a$; whence it appears that the greatest depth RS of the bend falls in the middle of the height.

BOOK II.

ON MOTION.

SECTION I.

ON THE MOTION OF A POINT.

CHAP. I.

ON EQUABLE AND ON VARIABLE MOTION.

203. In *equable* motion the velocity continues constant: in *variable* it goes on increasing or lessening; in the former of which cases the motion is said to be *accelerated*, in the latter to be *retarded*. Equable motion is caused by a force which, after having communicated to the moveable body a determinate velocity, ceases to act upon it; the body, by its inertia (7.) preserving the velocity impressed on it. Variable motion is caused by a force which, acting continually upon the moveable body, goes on at every instant increasing or diminishing its velocity. Such a force is called *accelerating* or *retarding*, and is measured at each instant by the ratio of that very small increment or decrement of velocity, which it causes in the moveable body, to that very small time in which the increment or decrement is caused.

204. If this ratio be constant, so that in equal times equal degrees of velocity are added to, or taken from, the moveable body, the accelerating, or retarding, force is *constant*, and the motion derived from it is said to be *equally accelerated* or *retarded*.

From what has been said the following propositions are easily deduced.

205. ***Proposition I.*** In equable motion, the velocity being constant, the spaces past over are proportional to the times in which they are past over.

206. *Proposition II.* In the case of a variable motion, calling ϕ the accelerating force, u the velocity, s the space, and t the time, the relation between these quantities is determined by the two equations

$$u = \frac{ds}{dt};\ \phi = \frac{du}{dt}.$$

207. *Coroll.* 1. From these two equations eliminating first t, and then u, are deduced the two other equations

$$\phi . ds = u du;\ \phi . dt = \frac{ds}{dt},$$

which, being contained in the two first, do not afford any new determination.

208. *Coroll.* 2. Having, then, two equations between the four indeterminate quantities ϕ, u, s, t, if any two of them be given, the other two will be known; and if one only be given, the relation, which subsists between any two of the others, will be known.

209. *Coroll.* 3. The preceding equations relate immediately to an accelerated motion; they are adapted to a retarded motion by making du negative.

CHAP. II.

ON MOTION EQUABLY ACCELERATED OR RETARDED.

210. ***Proposition I.*** In the equably accelerated motion of a body moving from a state of rest, and acted upon by a constant accelerating force g, the following equations obtain:

$$u = gt\,;\ s = \frac{u^2}{2g};\ s = \frac{gt^2}{2},$$

the last of which is contained in the two first.

For, if $\phi = g$, the equation (206.), $\phi\, dt = du$, integrated, becomes $u = gt$; also the other equation, $\phi\, ds = u\, du$, becomes, in like manner, $gs = \frac{u^2}{2}$; and, lastly, from the two first equations, eliminating u, there results the third $s = \frac{gt^2}{2}$. Nor is it necessary to correct the integrations by the addition of constant quantities, because, according to the hypothesis, u, s, t, vanish simultaneously.

211. *Coroll.* 1. The velocities, therefore, increase as the times; and the spaces as the squares of the times or of the velocities.

212. *Coroll.* 2. The spaces described in equal and successive intervals of time, reckoned from the beginning of motion, are in the ratio of the odd numbers 1, 3, 5, 7, &c.

213. *Coroll.* 3. If the moving body has, in the time t, described the space s with an equably accelerated motion, and has acquired the velocity u; and if then for an equal time t it it continues to move with an equable motion and with the acquired velocity u, it will describe the space $2s$.

214. *Coroll.* 4. The measure of the constant accelerating force g is obtained, when the space, which the moveable body has described in a given time t, is known. For $g = \frac{2s}{t^2}$.

215. *Coroll.* 5. The above-mentioned equations hold, when the body moves from a state of rest, as was laid down; but if it has a velocity c already impressed, this must be taken into account by determining constant quantities to be added to the integrals of $\phi dt = du$, and of $\phi ds = u\,du$. Thus, instead of the first three equations, we shall have the three following;

$$u = c + gt;\ s = \frac{u^2 - c^2}{2g};\ s = ct + \frac{gt^2}{2}.$$

216. *Proposition II.* In an equably retarded motion, if c be the initial velocity of the body moving in a direction contrary to that of the retarding force, the following equations obtain;

$$u = c - gt;\ s = \frac{c^2 - u^2}{2g};\ s = ct - \frac{gt^2}{2}.$$

These equations are readily obtained from the integration of the general equations (209.) $\phi dt = -du$; $\phi ds = -u\,du$.

217. *Coroll.* 1. The duration of this motion, and the space past over until the extinction of the impressed velocity c, are expressed as follows;

$$t = \frac{c}{g};\ s = \frac{c^2}{2g}.$$

218. *Coroll.* 2. This time, and this space, are (210.) precisely the same, as those in which the body, moving from a state of rest, with an equably accelerated motion, would acquire the velocity c. Wherefore, if, after the impressed velocity has been destroyed, the accelerating force continue to act, the body will pursue the same track, and will return, in an equal time, to the point whence it set out; and, when it has arrived there, it will have acquired, in an opposite direction, the initial velocity c.

219. *Coroll.* 3. In an equably retarded motion the velocities, at any points, are as the times to the cessation of motion; and the spaces which remain to be described in those times, are as the squares of those times or of the velocities.

CHAP. III.

ON THE VERTICAL MOTION OF HEAVY BODIES.

220. *Proposition.* The motion of heavy bodies, in their free vertical descent, is an equably accelerated motion; and in their ascent, it is an equably retarded motion.

This is evident from gravity being (58.) a constant force, and continually applied in a vertical direction.

221. *Coroll.* 1. If it be assumed that a heavy body, falling freely from rest, describes, in 1″, the space 16.09 English feet, or 4.9044 French metres, then, taking 1″ for the unity of time, and a metre for the unity of space, we shall have (214.) for the expression of this accelerating force $g = 9.8088$.

222. *Coroll.* 2. If, then, it be required to find the velocity acquired, or the space described, by a heavy body falling for a given time t, or conversely, the following forms must be used,

$$u = 9.8088\,t;\quad t = 0.1019\,u;$$
$$s = 4.9044\,t^2;\quad t = 0.4515\sqrt{s}.$$

223. *Coroll.* 3. If it be required to find the velocity acquired by a heavy body falling through a given space s, or, as it is called, the *velocity due to the altitude* s; or, conversely, the velocity being given, to find the *space due to it*, the formulas are

$$u = 4.4292\sqrt{s};\quad s = 0.0510\,u^2.$$

224. *Coroll.* 4. A body projected directly upwards, with a velocity c, will ascend, in the time $0.1019\,c$, to the height $0.0510\,c^2$. Thence, it will begin to descend; and, in a time equal to that of its ascent, it will return to the point whence it was projected, having acquired the velocity of projection c.

225. Scholium. In this, and in the following Chapters, the body is considered as if it were an atom or material point, no regard being had to mass or figure. In its proper place, we shall see that these latter affections do not change the free motion produced by gravity.

CHAP. IV.

ON THE VERTICAL MOTION OF HEAVY BODIES IN RESISTING MEDIUMS.

226. HYPOTHESIS. THE resistance of the air, and of every other fluid medium, in like circumstances, is proportional to the square of the velocity of the body which moves in it.

This retarding force may, therefore, be expressed by u^2 multiplied by a constant coefficient, which, for the convenience of calculation, we shall call gk^2; so that the retarding force is gk^2u^2. The constant coefficient depends upon the figure of the body, and upon the ratio of its specific gravity to that of the fluid: whence, the value of k is the same for the same body and for the same fluid, but changes in different bodies and in different mediums, according to laws which are taught in the science of Hydraulics.

227. *PROPOSITION I.* If a heavy body descend from rest, through a resisting medium, to find the relation between the space, the time, and the velocity.

Here the accelerating force is $g-gk^2u^2$. Wherefore, making $\phi = g(1-k^2u^2)$, the equations $\phi\,dt = du$, and $\phi\,ds = u\,du$, become

$$g\,dt = \frac{du}{1-k^2u^2};\quad g\,ds = \frac{u\,du}{1-k^2u^2}.$$

If these two equations be integrated, by determining the necessary constant quantities, upon the supposition that when $t=0$, then $s=0$, $u=0$, there result the two equations

$$u = \frac{1}{k}\cdot\frac{e^{2gkt}-1}{e^{2gkt}+1};\quad s = \frac{1}{2gk^2}\cdot\log.\frac{1}{1-k^2u^2}:$$

whence, eliminating u, we have the third equation

$$s = \frac{1}{gk^2}\cdot\log.\tfrac{1}{2}\{e^{gkt}+e^{-gkt}\}.$$

228. *Coroll.* The velocity of the descending heavy body can never exceed, nor even reach, the limit $u = \frac{1}{k}$. It approaches, however, indefinitely near to this limit, and the motion becomes very nearly uniform after a certain time, which is the shorter, in proportion as the coefficient k of resistance is greater.

229. *Proposition II.* If a heavy body ascend vertically through a resisting medium, projected with a given velocity c, to find the relation between the space, the time and the velocity.

Here, the retarding force is $g + gk^2u^2$; whence, making $\phi = g(1 + k^2u^2)$ the equations $\phi dt = -du$, $\phi ds = -udu$, become

$$gdt = \frac{-du}{1 + k^2u^2};\ gds = \frac{-udu}{1 + k^2u^2}.$$

Hence, integrating and determining the constant quantities, so that when $t = 0$, then $s = 0$, $u = c$, it will be found that

$$u = \frac{1}{k} \cdot \frac{kc - \tan. gkt}{1 + kc \tan. gkt};\ s = \frac{1}{2gk^2} \cdot \log. \frac{1 + k^2c^2}{1 + k^2u^2};$$

and eliminating u

$$s = \frac{1}{gk^2} \log (\cos. gkt + kc \sin. gkt).$$

230. *Coroll.* The duration of this motion and the height of the ascent are obtained, by putting, in the preceding equations, $u = 0$; whence

$$t = \frac{1}{gk} \text{ arc tan. } kc;\ s = \frac{1}{2gk^2} \log. (1 + k^2c^2).$$

CHAP. V.

ON CURVILINEAR MOTION.

231. *Proposition.* To determine the motion of a point acted upon by more than one force.

Let the position of the moveable point be referred to three co-ordinates x, y, z, and let the acting forces be reduced to three P, Q, R, parallel to these co-ordinates. Then, ds being the small arc, which the point will describe in the time dt, its velocity will be $u = \frac{ds}{dt}$. And because (19.) ds is the diagonal of a parallelepiped, which has for its sides dx, dy, dz, the moveable point cannot, in the instant dt, describe the small arc ds, without at the same time describing, in the directions of the ordinates x, y, z, the small spaces dx, dy, dz; for, instead of the velocity $\frac{ds}{dt}$, it may be conceived to move with the three velocities $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, parallel to the ordinates x, y, z. Hence the equation (207.) $\phi dt = d.\frac{ds}{dt}$; will give these three

$$P dt = d.\frac{dx}{dt}; \quad Q dt = d.\frac{dy}{dt}; \quad R dt = d.\frac{dz}{dt}.$$

232. *Coroll.* 1. These equations contain the complete determination of the motion. For, 1st, integrating, we shall have the partial velocities $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, from the composition of which results the absolute velocity of the point; 2dly, integrating afresh, we shall have the ordinates x, y, z, expressed in terms of

t, and thus we shall know the place of the point at any instant; 3dly, eliminating t, there will remain two equations between x, y, z; which will be those of the *trajectory*, or curve described by the moveable point.

233. *Coroll.* 2. If the co-ordinates be at right angles to each other, then $u\,du = P\,dx + Q\,dy + R\,dz$.

For, considering the element dt as constant, the equations (231.) of the motion are,

$$P = \frac{ddx}{dt^2};\quad Q = \frac{ddy}{dt^2};\quad R = \frac{ddz}{dt^2}:$$

multiply them by dx, dy, dz, respectively, and take their sum, and

$$P\,dx + Q\,dy + R\,dz = \frac{dx\,ddx + dy\,ddy + dz\,ddz}{dt^2}:$$

but, the ordinates being rectangular,

$$dx^2 + dy^2 + dz^2 = ds^2;$$

whence, $dx\,ddx + dy\,ddy + dz\,ddz = ds\,dds$,

and $P\,dx + Q\,dy + R\,dz = \dfrac{ds\,dds}{dt^2} = u\,du.$

234. *Coroll.* 3. If the forces, which act on the moveable point, be all in one plane, two ordinates x, y, will suffice; the trajectory will lie wholly in that plane, and its equation will be obtained, by eliminating t, from the two equations

$$P\,dt = d.\frac{dx}{dt};\quad Q\,dt = d.\frac{dy}{dt}.$$

CHAP. VI.

ON HEAVY BODIES PROJECTED OBLIQUELY.

235. ***Proposition.*** If a heavy body be projected, obliquely to the vertical line, with a velocity due to the altitude H, and if the abscissa x be taken in the vertical line drawn through the point of projection, and the ordinate y parallel to the direction of projection, the equation of the curve described by the projectile is $y^2 = 4Hx$.

Resuming the general equations (231.)

$$P\,dt = d.\frac{dx}{dt},\quad Q\,dt = d.\frac{dy}{dt},$$

since $P = g$, $Q = 0$, we shall have at the first integration,

$$\frac{dx}{dt} = A + gt;\quad \frac{dy}{dt} = B.$$

To determine the constant quantities, it must be observed that when $t = 0$, the velocity in the direction of $x = 0$, and the velocity in the direction of y is that due to the altitude H, or it is $= \sqrt{2gH}$. Substituting these values, and repeating the integration, we have

$$x = \tfrac{1}{2}gt^2;\quad y = t\sqrt{2gH},$$

without the need of any new constant quantities; because $t = 0$, gives $x = 0$, $y = 0$: and, eliminating t, $y^2 = 4Hx$.

236. *Coroll.* 1. The trajectory of heavy bodies, projected obliquely, is a parabola, having for its diameter the vertical drawn through the point of projection; and the parameter of this parabola is the quadruple of the height due to the velocity of projection.

237. *Coroll.* 2. Let the co-ordinates be changed, and instead of taking AP (Fig. 24.) for the abscissa x, and PM for the ordinate y, assume the horizontal straight line $AQ = z$ for the abscissa, and the vertical $QM = u$ for the ordinate: let the *angle of elevation*, or of projection, $TAQ = f$; then if the horizontal straight line PR be drawn through P meeting MQ produced in R,

$$MR = PM \sin. f; \quad PR = PM \cos. f;$$

or, $\quad u + x = y \sin. f; \quad z = y \cos. f.$

Whence, $\quad u = z \tan. f - x.$

But (235.) $x = \dfrac{y^2}{4H} = \dfrac{z^2}{4H \cos.^2 f}$; therefore, lastly,

$$u = z \tan. f - \frac{z^2}{4 . H \cos.^2 f}.$$

238. *Coroll.* 3. The greatest value of u belongs to the vertex of the parabola, indicating the greatest height to which the projectile can rise. It corresponds to the point where

$$z = H . \sin. 2f; \quad u = H . \sin.^2 f.$$

Hence it may readily be found, either by calculation or by geometrical construction.

239. *Coroll.* 4. Calling the horizontal reach, or the *range*, $AB = A$,

$$A = 2 H \sin. 2f.$$

The range is greatest when the angle of projection is half a right angle, and then $A = 2H$; also the ranges are equal, at two angles of projection, the one of which exceeds half a right angle as much as the other falls short of it.

240. *Coroll.* 5. Two of the three quantities A, H, f, being given, the third may be found; and, more generally, three of the four quantities u, z, H, f, being given, the fourth may be found by means of the equations of Art. 237.

Let it be required, for example, to find what must be the elevation of the projection, in order that the projectile thrown with a given velocity, may hit a given mark, the co-ordinates of which are $z = a$, $u = b$.

We shall have, (237.)

$$b = a \tan. f - \frac{a^2}{4 \, . \, H \cos.^2 f}.$$

But $\frac{1}{\cos.^2 f} = 1 + \tan.^2 f$; substituting this value and resolving the quadratic equation, we obtain

$$\tan. f = \frac{2\,H \pm \sqrt{(4\,H^2 - 4\,b\,H - a^2)}}{a}.$$

Generally speaking, therefore, a given mark may be hit by two different throws; but if $4\,b\,H + a^2 > 4\,H^2$, both the values become imaginary, and the problem is impossible.

CHAP. VII.

ON THE PATH OF PROJECTILES IN THE AIR.

241. LET $g R$ express the resistance of the medium. This force being exerted in the direction of the element ds of the curve described by the projectile, if the curve be referred to an horizontal axis drawn through the point of projection, if the horizontal abscissas be called x, the vertical ordinates y; and if the resistance $g R$ be resolved into two forces, the one parallel to x, the other to y; the first shall be $g R \frac{dx}{ds}$, and the second $g R \frac{dy}{ds}$. The heavy body will therefore be acted upon in the direction of x by the force $-g R \frac{dx}{ds}$, and then in the direction of y by the force $-g - g R \frac{dy}{ds}$. Thus, the investigation of the trajectory will be reduced to the general problem of Art. 231.

242. *Proposition.* To determine the trajectory of a heavy body in a resisting medium.

From Art. 231. we have the equations

$$-g R \frac{dx}{ds} dt = d. \frac{dx}{dt},$$

$$-g dt - g R \frac{dy}{ds} dt = d. \frac{dy}{dt},$$

from which t is to be eliminated. Differentiating the second member of both equations, on the supposition that dx is constant, and substituting in the second, from the first equation, the value of ddt, there will result

$$ds\,ddt = gRdt^3;\ ddy = -gdt^2:$$

And, differentiating the last equation,

$$d^3y = -2gdtddt.$$

It is easy to eliminate from these equations the elements dt, ddt; whence

$$2R(ddy)^2 + dsd^3y = 0,$$

which is the equation of the trajectory. Hence the law of resistance being given, the curve described by the projectile will be known, and conversely.

243. *Coroll.* 1. If the resistance be proportional to the square of the velocity, put $gR = gk^2u^2 = gk^2\frac{ds^2}{dt^2}$.

$$\text{Now } dt^2 = -\frac{ddy}{g};$$

$$\text{therefore } R = -gk^2\frac{ds^2}{ddy};$$

whence, the equation of the trajectory becomes

$$\frac{d^3y}{ddy} = 2gk^2ds;$$

and, integrating,

$$\frac{ddy}{dx^2} = Ae^{2gk^2s}.$$

To determine the constant quantity A, for ddy put its value $-gdt^2$ (242.); and

$$\frac{dx^2}{dt^2} = -\frac{g}{Ae^{2gk^2s}}.$$

But, if f be the angle of elevation, and H the space due to the velocity of projection, at the beginning of motion, when $s=0$, the horizontal velocity $\frac{dx}{dt} = \cos. f\sqrt{2gH}$.

$$\text{Hence } A = -\frac{1}{2H\cos.^2 f};$$

and the equation becomes

$$\frac{ddy}{dx^2} = - \frac{e^{2gk^2s}}{2\,H \cos.^2 f}.$$

244. *Coroll.* 2. A first integral of this equation may be easily obtained. Put, for abridgement, $2gk^2 = h$, and $\frac{dy}{dx} = p$; and the equation will become

$$\frac{dp}{dx} = \frac{e^{hs}}{2\,H \cos.^2 f}.$$

Multiplying this by the identical equation $dx\ \surd(1 + p^2) = ds$, and then integrating, we shall have

$$p\ \surd(1 + p^2) + \log.\ \{p + \surd(1 + p^2)\} = C - \frac{e^{hs}}{Hh \cos.^2 f}.$$

The constant quantity C is determined, by considering that when $s = 0$, then $p = \tan. f$.

It is not possible, however, to proceed further with the integration; and it therefore becomes necessary to search for expedients, from which we may know, by an approximation, the course of the trajectory.

CHAP. VIII.

A METHOD OF DESCRIBING, BY APPROXIMATION, THE TRAJECTORY OF A HEAVY BODY PROJECTED OBLIQUELY IN THE AIR.

245. Eliminating s from the equations of the preceding article, and for the sake of brevity, putting

$$p\sqrt{(1+p^2)} + \log.\{p + \sqrt{(1+p^2)}\} = P,$$

we shall have $dx = \dfrac{dp}{Ph - Ch}$;

whence $dy = p\,dx = \dfrac{p\,dp}{Ph - Ch}$;

wherefore, $x = \displaystyle\int \frac{dp}{Ph - Ch}$; $y = \displaystyle\int \frac{p\,dp}{Ph - Ch}$.

If these formulas could be integrated, we should have x and y in terms of p; and, eliminating p, we should have the equation of the trajectory. But, since these formulas cannot be integrated in finite terms, it will avail us to follow the method taught by Euler; so as to obtain the integrals by approximation. As this method may be useful, as well here as in other physico-mathematical problems, the application of it to this case deserves to be shewn.

246. Lemma. If $y = \int X\,dx$; if when $x=a$, then $y=b$; if when $x = a$, $a + \delta$, $a + 2\delta$, $a + 3\delta \ldots . a + n\delta$, then $X = A, A', A'', A''' \ldots . A^{(n)}$; and if the difference δ be very small, the value of the integral $\int X\,dx$, from $x = a$, to $x = a + n\delta$, will be $y = b + \delta\{\tfrac{1}{2}A + A' + A'' + A''' \ldots . + \tfrac{1}{2}A^{(n)}$.

In the very small interval between $x = a$, and $x = a + \delta$, it may be assumed that the function X remains constant. And because at the beginning of this interval $X = A$, and at the end, $X = A'$, we may assume for this constant value the arithmetic mean $X = \frac{1}{2}A + \frac{1}{2}A'$. Wherefore, for this interval, we shall have $Xdx = \delta(\frac{1}{2}A + \frac{1}{2}A')$; and the value of y from $x = a$, to $x = a + \delta$, will be $y = b + \delta(\frac{1}{2}A + \frac{1}{2}A')$.

For the next interval, comprised between $x = a + \delta$ and $x = a + 2\delta$, we may similarly consider $X = \frac{1}{2}A' + \frac{1}{2}A''$; whence $Xdx = \delta\ (\frac{1}{2}A' + \frac{1}{2}A'')$. Adding this value of Xdx to the preceding, it appears that the value of y from $x = a$, to $x = a + 2\delta$, is $y = b + \delta\ (\frac{1}{2}A + A' + \frac{1}{2}A'')$: and so on, to the last.

247. ***Proposition I.*** To construct the trajectory of a heavy body projected obliquely in the air.

Applying the Lemma (246.) to the integrals of Art. 245, and beginning with $x = \int \frac{dp}{Ph - Ch}$, we know that when $p = \tan. f$, then $x = 0$. Now since the tangent p of the angle, which the curve makes with the direction of x, becomes gradually less and less, until at the vertex of the curve it is nothing, and afterwards becomes negative, and goes on increasing in the descending branch, make, successively, $p = \tan. f$, $\tan f - \delta$, $\tan. f - 2\delta$, $\tan. f - 3\delta$, &c. until $p = 0$; and then $p = -\delta, -2\delta, -3\delta$, &c. ad infinitum; taking for δ any number at pleasure; and the smaller it is taken the more exact will be the description of the curve. Calculate the values A, A', A'', A''', &c. of the quantity $\frac{1}{Ph - Ch}$, when p becomes successively $\tan. f$, $\tan. f - \delta$, &c. Then, putting successively $n = 1, 2, 3$, &c. ad infinitum, we shall have (246.) the values of x which correspond to the angles where $p = \tan. f - \delta$, $\tan. f - 2\delta$, &c. ad infinitum.

Also, by the other equation $y = \int \frac{pdp}{Ph - Ch}$, the values may be found, which $\frac{p}{Ph - Ch}$ acquires, when p becomes $\tan. f$, $\tan. f - \delta$, &c.: and proceeding in like manner, the values of y will be known, which correspond to the same angles, where $p = \tan. f - \delta$, $\tan, f - 2\delta$, &c.

Wherefore, we shall have the abscissas x, and the corresponding ordinates y, of all the points of the curve, in which p becomes successively tan. $f - \delta$, tan. $f - 2\delta$, &c. We can, therefore, mark out the series of points through which the trajectory must pass, and thus we can trace its course.

248. *Coroll.* 1. The ordinate y which is found to correspond to $p = 0$, will be the altitude of the projection; and the corresponding abscissa x will be the amplitude of the ascending branch. Pursuing the calculation to where $y = 0$, the corresponding abscissa will be the total amplitude of the projection; and the corresponding value of p will shew the angle at which the descending branch of the curve cuts the axis.

249. *Coroll.* 2. Beyond that point, the ordinate y becomes negative and increases indefinitely: not so the abscissa x, which has a limit, that it does not pass; whence we may conclude that the descending branch of the curve has a vertical assymptote.

For, when p is negative, we have

$$P = -p\sqrt{(1+p^2)} + \log.\{-p + \sqrt{(1+p^2)}\}.$$

$$\text{But } -p + \sqrt{(1+p^2)} = \frac{1}{p + \sqrt{(1+p^2)}};$$

$$\text{therefore } P = -p\sqrt{(1+p^2)} - \log.\{p + \sqrt{(1 p^2)}\}.$$

And if p, besides being negative, be very great, we may assume $\sqrt{(1+p^2)} = p$, and the logarithm may be neglected; because the logarithm is inconsiderable when compared with the number itself. When, therefore, p is negative and very great, $P = -p^2$; and since the constant quantity C may also then be neglected, in comparison with p, we shall have

$$\frac{1}{Ph - Ch} = -\frac{1}{hp^2}; \text{ and } \frac{p}{Ph - Ch} = \frac{1}{hp}.$$

Whence $dx = -\dfrac{dp}{hp^2}$, and $x = \text{const.} + \dfrac{1}{hp}$; which value does not increase indefinitely.

But with respect to y, since $dy = \dfrac{dp}{hp}$, $y = \text{const.} + \dfrac{1}{h}\log. p$; which value increases indefinitely along with p.

250. *Proposition II.* To construct the trajectory of a heavy body projected obliquely in the air, when the angle of elevation

is very small. The labour of constructing the curve by approximation becomes in this case more simple; for since the curve deviates very little from the axis, the abscissa x may be put for the arc s: whence

$$(244.)\ dp = -\frac{e^{hx}dx}{2\ H \cos.^2 f}.$$

Integrating, so that $x = 0$ may give $p = \tan. f$, there results

$$p = \frac{dy}{dx} = \tan. f + \frac{1 - e^{hx}}{2\ Hh \cos.^2 f}.$$

And integrating afresh, so that $x = 0$ may give $y = 0$,

$$y = x \tan. f + \frac{1 + hx - e^{hx}}{2 Hh \cos.^2 f};$$

whence, for any abscissa x, the corresponding ordinate may be computed.

251. *Coroll.* Putting this value of y equal to nothing, the resulting equation, besides $x = 0$, will give another value of x, which will be the amplitude of the projection. If, likewise, the value of p be made equal to nothing, the resulting equation will give the abscissa x, which corresponds to the greatest ordinate, that is, the amplitude of the ascending branch: and putting this value of x in that of y, we shall have the greatest ordinate, or the altitude of the projection.

CHAP. IX.

ON THE MOTION OF A POINT ON A GIVEN CURVE, OR ON A GIVEN SURFACE.

252. *Proposition I.* To determine the motion of a point, or elementary particle of matter, compelled to run over a given surface.

Let the equation of the surface, referred to three rectangular axes, be $l dx + m dy + n dz = 0$; let an elementary particle of matter move upon it, acted upon by the forces P, Q, R, parallel to the axes; and let K be the pressure which the particle exerts upon the subjacent surface. This pressure cannot but be in a direction perpendicular to the surface, for, if it were oblique, it might be resolved into two, the one normal, the other tangential; and this latter would not exert a pressure; which is contrary to the hypothesis. Resolving then the pressure K into three forces parallel to the three axes, these will be (30. 101.)

$$\frac{lK}{M},\ \frac{mK}{M},\ \frac{nK}{M},$$

putting, for the sake of brevity, $M = \sqrt{(l^2 + m^2 + n^2)}$.

Now the moveable point will remain in the same state, if we suppose the subjacent surface to be removed, and, in its place, a force to be applied equal and contrary to the pressure K. We may, therefore, assume that the point moves freely, acted upon by the three forces $P - \frac{lK}{M}$, $Q - \frac{mK}{M}$, $R - \frac{nK}{M}$, parallel to the three rectangular axes. And, therefore, considering the element dt as constant, we shall (231.) have the three equations

$$P - \frac{lK}{M} = \frac{ddx}{dt^2};\quad Q - \frac{mK}{M} = \frac{ddy}{dt^2};\quad R - \frac{nK}{M} = \frac{ddz}{dt^2}.$$

253. *Coroll.* 1. By these equations the motion of the point is fully determined, and the pressure which it exerts against the surface.

For, first, calling u the velocity, if we multiply the first equation by dx, the second by dy, the third by dz, and then take their sum, making $ldx + mdy + ndz = 0$, we shall find, as in Art. 233.

$$udu = Pdx + Qdy + Rdz.$$

Secondly, multiplying similarly the equations by l, m, n, and observing that $dt = \frac{ds}{u}$, their sum will give

$$K = \frac{lP + mQ + nR}{M} - \frac{u^2}{M} \cdot \frac{lddx + mddy + nddz}{ds^2}.$$

Thirdly, eliminating K, there will remain two equations between the variable quantities x, y, z, t; to which, if there be added the given equation of the surface, we shall have three equations, by means of which the co-ordinates x, y, z, are to be expressed in terms of t.

Lastly, eliminating t, the two equations which remain between x, y, z, will determine the trajectory.

254. *Coroll.* 2. If the moveable point is not acted upon by any accelerating force, its motion is uniform; and whatever be the curve, which it describes, it retains, without alteration, the impressed velocity.

For, if $P = Q = R = 0$, then $udu = 0$, and therefore, $du = 0$, and u is constant.

255. *Coroll.* 3. If the moveable point be acted upon by gravity alone, through whatever curve it descends, or ascends, it will have at each point, the same velocity which it would have had, if it had descended, or ascended, vertically, through an equal altitude.

For, supposing the axis of x to be vertical,

$P = \pm g$; $Q = R = 0$; whence $udu = \pm gdx$. Wherefore, &c.

256. *Proposition II.* To determine the motion of a point compelled to describe a curve lying all in one plane.

This problem is contained in the preceding. If the curve be referred to two axes, at right angles to each other, in its plane, we shall have the equation $l\,dx + m\,dy = 0$. The directions of the forces acting upon the moveable point must needs be in the same plane, otherwise the point would leave the plane. They may, therefore, be reduced to two P and Q, parallel to x and y, respectively. And, putting $M = \sqrt{(l^2 + m^2)}$ we shall have (252.)

$$P - \frac{lK}{M} = \frac{ddx}{dt^2}; \quad Q - \frac{mK}{M} = \frac{ddy}{dt^2}.$$

257. *Coroll.* 1. The pressure which the moveable point exerts against the curve, will be (253.)

$$K = \frac{lP + mQ}{M} - \frac{u^2}{M} \cdot \frac{lddx + mddy}{ds^2};$$

and, because $m = \frac{-l\,dx}{dy}$, $M = \frac{l\,dx}{dy}$, and the radius of curvature $r = \frac{\mp ds^3}{dx\,ddy - dy\,ddx}$, substituting these values, we have

$$K = \frac{P\,dy - Q\,dx}{ds} \mp \frac{u^2}{r}.$$

The upper sign is to be taken when the curve is concave towards the axis, the lower when it is convex.

258. *Coroll.* 2. Analysing this expression for K, it appears that the first term $\frac{P\,dy - Q\,dx}{ds}$ is no other than (34.) the force which arises from resolving each of the forces P, Q, into two, one in the direction of a tangent, the other in that of a normal, to the curve, and taking the sum of the normal forces. And, from the construction made in Art. 34, it is plain that the direction of this normal force makes an acute angle with the direction of x, whilst x is positive. Wherefore, wherever the value of K is positive, there the direction of the pressure will make an acute angle with the direction of x taken positively; and where it is negative, it will make the angle obtuse. By this we can always know, when the pressure urges the body against the curve, and when it lifts the body from it, forcing it to abandon the path.

259. *Coroll.* 3. K being the aggregate of the two terms $\frac{P\,dy - Q\,dx}{ds}$ and $\frac{u^2}{v}$, the pressure expressed by the first term

originates, as we have said, from that portion of the acting forces, which pressing perpendicularly against the curve, is counteracted by the resistance of the curve. But the pressure expressed by the second term is independent of the acting forces, and takes place even when there is no acting force at all, and when the body moves only in consequence of its previously acquired velocity. This pressure is distinguished by the name of *Centrifugal Force*, and originates in the inertia of the moveable body; as we shall soon explain.

260. SCHOLIUM. Let a moveable point descend through the curve AMB (Fig. 25.) and in the time dt, let it have described the space Mm. Arrived at m, it tends, by its inertia, to describe, in the time dt, the tangent straight line $mq = Mm$. As the resistance of the curve compels it to pass through the element $M\mu$, so resolving the virtual velocity mq into two $m\mu$, mp, this last is counteracted by the curve, and therefore the moveable point presses it perpendicularly with a force corresponding to this extinguished velocity; and this is the centrifugal force.

Let us then express its value analytically.

This force, in the instant dt, would cause the moveable point to describe the small space mp; it may be considered as a constant force, because it is supposed to act only for the single instant dt. Wherefore, from the equation (214.) $g = \frac{2s}{t^2}$, it is evident that this force is $\frac{2\,mp}{dt^2}$.

Let now r be the radius of curvature of the curve AMB, at the point m. The arc $Mm\mu$ may be considered as a circular arc of which r is the radius: and, by the property of the circle, mq is a mean proportional between mp and $mp + 2r$, or, between mp and $2r$. Therefore, $mp = \frac{\overline{mq}^2}{2r}$. But $mq = Mm = ds = u.dt$. Wherefore, $mp = \frac{u^2.dt^2}{2r}$. By substituting this value, the centrifugal force is found to be expressed by $\frac{u^2}{r}$, exactly as was concluded, by another train of reasoning, in the preceding Article.

CHAP. X.

ON THE DESCENT OF HEAVY BODIES DOWN INCLINED PLANES.

261. ***Proposition I.*** A HEAVY body, placed on an inclined plane, will descend along the straight line drawn from the body perpendicular to the common section of the plane and the horizon.

Let the heavy body be placed at A (Fig. 26.) on the inclined plane I; let RS be the common section of this plane, and of the horizontal plane O; and in I let there be drawn IB, and in O, BC, both perpendicular to RS: also from A let there be drawn AC perpendicular to BC. The plane ABC is perpendicular to the plane O, and is therefore vertical: it is likewise perpendicular to the plane I. Resolving, then, gravity, which acts in the direction of the vertical AC, into two forces, the one in the direction of AB, the other perpendicular to the plane I, this latter will be counteracted, and the body, urged by the other force, will begin to move along AB.

And because this reasoning is applicable not only to the point A, but to every other point of AB, it is manifest that the heavy body having set out along AB, will continue to descend through it.

262. *Coroll.* If the heavy body be not simply placed on the plane I, but projected along it in any direction AT, it will describe, in the plane, the parabola AQ, which will have AT for its tangent, AB for its diameter, and the quadruple of the height due to the velocity of projection, for its parameter.

263. *Proposition II.* If the plane I be inclined to the vertical plane at the angle $BAC = m$, the force which acts upon the heavy body descending through AB, will $= g \cos. m$: and the force with which the heavy body presses the plane itself, will $= g . \sin. m$.

264. *Coroll.* 1. If the length *AB* of the plane, represent gravity, the altitude *AC* will represent that part of it which urges the heavy body down the plane, and the base *BC* will represent the other part, which operates in pressing against the plane.

265. *Coroll.* 2. The motion of descent is equally accelerated, and is determined by the equations (210.) $u = gt \cos. m$; $s = \frac{u^2}{2g . \cos. m}$; $s = \frac{qt^2 \cos. m}{2}$. But, if the body ascend up the plane, in consequence of an impressed velocity, the motion will be equally retarded, and will be in like manner determined by the equations (216.), putting every where $g \cos. m$ for g.

266. *Coroll.* 3. The velocities acquired, and the spaces past over, in equal times, by two heavy bodies falling, the one through the vertical line, the other through the inclined plane, will be to one another as $1 : \cos. m$; or as the length to the height of the plane.

267. *Coroll.* 4. Hence, whilst the former descends through the whole height *AC*, the other will arrive at the point *D*, on which falls the perpendicular *CD* drawn from *C* to the plane.

268. *Coroll.* 5. All the chords of a circle, which terminate in either extremity of a vertical diameter, are described in the same time; that is, in the time in which a heavy body, falling freely, would describe the diameter.

CHAP. XI.

ON THE DESCENT OF HEAVY BODIES THROUGH A CYCLOID.

269. *Proposition I.* If the cycloid FAG (Fig. 27.) have its base FG horizontal, and if a heavy body placed in the point P, and left to itself, descend through the arc PMA, to find its velocity at any point M, and the pressure which it exerts against the curve.

From P and M draw PS, MR, parallel to FG, and meeting BA in S and R, respectively. Let $AR=x$, $RM=y$, $AM=s$; and let the diameter of the generating circle $AB=\frac{1}{2}a$, and the abscissa, of the point P, $AS=h$. The equation of the cycloid is $s.ds=a.dx$; whence

$$ds=\frac{dx\sqrt{a}}{\sqrt{2x}};\quad \frac{ds}{dy}=\sqrt{\frac{a-2x}{a}};$$

and the radius of curvature $r=\sqrt{(a^2-2ax)}$.

Wherefore, the velocity u, at M, will (255.) be that which is due to SR, and (210.) $u=\sqrt{2g.(h-x)}$.

Again, since $P=-g$, and $Q=0$, the pressure will be (257.)

$$K=-\frac{g.dy}{ds}-\frac{u^2}{r}=-\frac{g}{\sqrt{a}}.\frac{a+2h-4x}{\sqrt{(a-2x)}}.$$

And, this value being always negative, the pressure will (258.) make an obtuse angle with x, whilst x is positive, and therefore will act in the direction of MM', pressing the body against the curve.

270. *Coroll.* 1. The time of descent through the arc *PM* will be had from the equation

$$dt = -\frac{ds}{u} = \frac{\sqrt{a}}{2\sqrt{g}} \cdot \frac{dx}{\sqrt{(hx - x^2)}}.$$

Integrating so that $x = h$ may give $t = 0$, we shall have

$$t = \frac{\sqrt{a}}{2\sqrt{g}} \cdot \text{arc. cos.} \frac{2x - h}{h}.$$

271. *Coroll.* 2. Making $x = 0$, we shall have the whole time of descent from *P* to the lowest point *A*; and it will be $t = \frac{\pi\sqrt{a}}{2\sqrt{g}}$, π being the circumference of a circle of which the diameter is unity.

This value is altogether independent of the position of the point of departure *P*; whence follows that most singular property of the cycloid, that, at whatever point of its periphery the heavy body be left at liberty to fall down it, it will arrive at the lowest point in the same time. On account of this property the cycloid is called *tautochronal.*

272. *Coroll.* 3. The heavy body having arrived at the lowest point *A*, with the velocity due to *AS*, it will ascend through the opposite arc *AQ*. In each point of its ascent, its velocity will be the same as in the corresponding point of the descent, that is, as in the point which belongs to the same vertical abscissa. And when it has reached the point *Q*, of the same altitude with the point *P*, it will stop; and then falling through the arc *QA*, and re-ascending through *AP*, it will follow the same course, continually. The time of ascent through *AQ* will be $\frac{\pi\sqrt{a}}{2\sqrt{g}}$, the same as that of the descent through *PA*.

273. *Proposition II.* To find the line of quickest descent from a given point *A* (Fig. 28.) to another given point *B*.

Let AMB be the line of quickest descent; make the vertical line $AP=x$, and PM being drawn parallel to the horizontal line AL, let $PM=y$. Then the velocity at M will be $\sqrt{2gx}$, and the time of descent through AM will be represented by the integral $\int \frac{ds}{\sqrt{2gx}}$. This integral, taken from the point A to the point B, is to be a minimum. Wherefore, according to the theory of the Method of Variations, we must put $\delta \int \frac{ds}{\sqrt{2gx}}=0$; or, making $dy=p\,dx$, so that $ds=dx\sqrt{(1+p^2)}$ we must put $\delta \int \frac{dx\sqrt{(1+p^2)}}{\sqrt{2gx}}=0$. According to the rules of the Method of Variations, the equation of the required curve is found by differentiating $\frac{\sqrt{(1+p^2)}}{\sqrt{2gx}}$, and putting the coefficient of dp equal to a constant quantity. The equation will, therefore, be

$$\frac{p}{\sqrt{2gx(1+p^2)}}=C;$$

or, for the sake of brevity, making $\frac{1}{2gC^2}=k$,

$$dy=dx\sqrt{\frac{x}{k-x}},$$

the equation of a cycloid, which has its origin in the point A, its base in the horizontal line AL drawn through A, and the diameter of its generating circle equal to k.

If, therefore, the indefinite horizontal line AL be drawn, and upon it, as a base, there be described the cycloid AQL which passes through the given point B, that curve will be the line of quickest descent from A to B. On account of this property, the cycloid is called *brachystochronal.*

274. SCHOLIUM. To describe from the given point A, on the indefinite base AL, the cycloid, which shall pass through

the given point B, is a geometrical problem, of which the following is the most simple solution.

From the point A, describe on the base AL, any cycloid Aql, which has oq for the diameter of its generating circle. Join A, B, and let AB cut this cycloid Aql in b. Find a fourth proportional, OQ, to Ab, AB, and oq; and OQ shall be the diameter of the generating circle of the cycloid AQL, which passes through the given point B.

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CHAP. XII.

ON THE DESCENT OF HEAVY BODIES THROUGH CIRCULAR ARCS.

275. ***Proposition I.*** If FAG (Fig. 29.) be an arc of a vertical circle, through which a heavy body falls from the point P, to determine, at any point M, the velocity and the pressure.

Draw the vertical diameter aA, and the horizontal diameter FG; also draw PS and MR, parallel to FG; let $AR=x$, $RM=y$, $AM=s$; the radius $CA = a$, and the abscissa of the point of departure $AS = h$. From the nature of the circle we have $ds = \frac{a\,.\,dx}{\sqrt{(2ax-x^2)}}$; $\frac{dy}{ds} = \frac{a-x}{x}$; and the radius of curvature $r = a$.

The velocity at M will be due to SR, so that $u = \sqrt{2g(h-x)}$. And since $P = -g$, $Q = 0$, the pressure will be

$$K = -g\frac{dy}{ds} - \frac{u^2}{r} = -\frac{g}{a}(a + 2h - 3x).$$

This value is always negative, and shews that the pressure will always be exerted against the curve in the direction MM'.

276. *Coroll.* 1. To determine the time of descent, we have the equation

$$dt = -\frac{ds}{u} = -\frac{a}{\sqrt{2g}} \cdot \frac{dx}{\sqrt{(h-x)\,.\,(2ax-x^2)}},$$

which, integrated from $x = h$ to $x = 0$, will give the time of descent from P to A.

277. *Coroll.* 2. But this equation can only be integrated by means of series. A converging series is obtained by writing the equation in this form,

$$dt = -\frac{\sqrt{a}}{2\sqrt{g}} \cdot \frac{dx\left(1-\frac{x}{2a}\right)^{-\frac{1}{2}}}{\sqrt{(hx-x^2)}},$$

and developing the numerator by Newton's binomial theorem, which gives

$$\left(1-\frac{x}{2a}\right)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}\cdot\frac{x}{2a} + \frac{1.3}{2.4}\cdot\frac{x^2}{4a^2} + \frac{1.3.5}{2.4.6}\cdot\frac{x^3}{8a^3} + \&c.$$

Thus is dt expressed by an infinite series, all the terms of which are of the form $-\frac{A x^n dx}{\sqrt{(hx-x^2)}}$. By the known rules of the integral calculus, the integral of each of these terms is made to depend upon the integral of $\frac{-dx}{\sqrt{(hx-x^2)}}$, which taken from $x=h$, to $x=0$, is found to be equal to π. Thus we have

$$t = \frac{\pi\sqrt{a}}{2\sqrt{g}}\cdot\left\{1+\left(\frac{1}{2}\right)^2\frac{h}{2a}+\left(\frac{1.3}{2.4}\right)^2\cdot\frac{h^2}{4a^2}+\left(\frac{1.3.5}{2.4.6}\right)^2\cdot\frac{h}{8a^3}, \&c.\right\}.$$

278. *Coroll.* 3. If h be very small in comparison with a, much more will x be so; and then the differential equation (276.), neglecting x^2 in comparison with $2ax$, will become

$$dt = -\frac{\sqrt{a}}{2\sqrt{g}} \cdot \frac{dx}{\sqrt{(hx-x^2)}}$$

like that of Art. 270, which belongs to the cycloid.

Whence it appears that circular arcs, in which the sagitta is very small compared with the radius, or which are of very few degrees, have, in common with the cycloid, the singular property, that from whatever point of the arc a heavy body be let fall, it will arrive at the lowest point in the same time $t = \frac{\pi\sqrt{a}}{2\sqrt{g}}$.

279. *Proposition II.* If the same circular arc FAG be inverted, and if a heavy body, placed at the point p, descend over

the convex circumference, to find the velocity and the pressure at any point m.

The same construction being made as in the preceding proposition, the velocity u will be that which is due to sr; whence, if $ar = x$, and $as = h$, we shall have $u = \sqrt{2g(x-h)}$.

And since $P = g$, $Q = 0$, we shall have the pressure

$$K = \frac{g.dy}{ds} - \frac{u^2}{r} = \frac{g}{a}(a + 2h - 3x).$$

280. *Coroll.* Whilst $x < \frac{a+2h}{3}$, K is positive, and the pressure is exerted in the direction mm'. When $x = \frac{a+2h}{3}$, $K = 0$. Afterwards, when $x > \frac{a+2h}{3}$, K is negative, and the heavy body quits the curve.

The body will, therefore, go on describing the circular arc pt, to which belongs the abscissa $aq = \frac{a+2h}{3}$, so that $sq = \frac{a-h}{3} = \frac{1}{3}Cs$. Arriving at t, it will there escape from the curve, and describe a parabola which will have, at t, a common tangent with the circle, its diameter vertical, and its parameter $= 4\sqrt{2g.sq}$.

CHAP. XIII.

ON THE SIMPLE PENDULUM.

281. ***Proposition I.*** The oscillations of a heavy pendulum, which is subject to no extraneous resistance or impulse, go on for ever equal in length, and are performed in equal times.

From what has been said in Art. 218, 255, it appears that the heavy body will ascend, in an equal time, to a height equal to that whence it descended; it will then return through the same track, retracing, without end, the arc which it first described. And this will happen, whatever be the curve which the oscillating pendulum describes. We shall proceed to shew, separately, what happens to pendulums oscillating through arcs of a cycloid, and of a circle.

282. *Proposition II.* The oscillations through cycloidal arcs, of any breadth, are isochronal.

If we call $\frac{1}{2}a$ the diameter of the generating circle of the cycloid, the time of the middle of the oscillation, that is, of describing the arc PA (Fig. 27.) is $t = \frac{\pi\sqrt{a}}{2\sqrt{g}}$; and that of the whole oscillation, that is, the time of describing the arc PAQ is $t = \frac{\pi\sqrt{a}}{\sqrt{g}}$ (271.272.).

283. Scholium. The cycloid being the evolute of itself, a cycloidal pendulum is obtained by making the thread CA oscillate between two plates CF, CG, curved in the form of two equal semi-cycloids, similarly posited with respect to the semi-cycloids GA, FA. Whence it appears that the length of the thread CA ought to be the double of BA; and that $CA = a$.

In general, the pendulum may be made to describe any curve, by causing the thread to vibrate between two plates bent in the form of the evolute of that curve. If the thread be free, the pendulum will describe a circle.

284. *Proposition II.* The oscillations through circular arcs, of a small number of degrees, are isochronal, whatever be the breadth of the arc described; and they are synchronal with those of a cycloidal pendulum of equal length.

Calling a the radius of the circle, or the length of the pendulum, the time to the middle of the oscillation will be (278.) $t = \frac{\pi\sqrt{a}}{2\sqrt{g}}$, and that of the entire oscillation will be the double of it.

285. *Coroll.* Comparing together the oscillations of pendulums of different lengths, whether they be cycloidal, or circular, provided that small arcs of circles be described, the times of the oscillations are as the square roots of the lengths. And, consequently, the numbers of oscillations made in the same time, are inversely as the square roots of the lengths.

286. Scholium. The isochronism of oscillations through cycloidal arcs of any breadth, or through circular arcs of very small breadth, may also be proved without the aid of integration, in the following manner.

Let two arcs of different breadth, PA, pA, (Fig. 30.) be terminated in A, their lowest point. Suppose the arc PA to be divided into an indefinite number of elementary parts PM, and the arc pA into an equal number of such parts pm. If the accelerating forces with which the heavy body begins to describe the small homologous arcs PM, pm, be proportional to the arcs themselves, these arcs will be described in equal times. For, since (210.) $t^2 = \frac{2s}{g}$, it is clear that if g is proportional to s, t will be constant. And if this happens in each of the homologous points of the arcs PA, pA, it is evident that each of the elements of the first will be described in an equal time with the corresponding element of the second; and thus also the entire arcs PA, pa will be described in equal times.

Now, whether the PpA be a cycloid, or a circular arc of small breadth, it happens that the tangential accelerating forces, in any two homologous points P, p, are proportional to the arcs PA, pA, and therefore also to the elements PM, pm. For, let $AE=x$, $EP=y$, $AP=s$. The tangential force in P will (34.) be $-\frac{g\,dx}{ds}$. And, if PpA be a cycloid, we have $s^2=2ax$, and therefore, $\frac{dx}{ds}=\frac{s}{a}$, and $\frac{g\,dx}{ds}=\frac{gs}{a}$: wherefore the tangential force at P is proportional to the arc PA.

But if PpA be an arc of a circle, we have $\frac{dx}{ds}=\frac{y}{a}$; hence $\frac{g\,dx}{ds}=\frac{gy}{a}$: where, if the arc be very small, the semi-chord y may be taken as equal to the arc s. Here, again, therefore, the tangential force at P is proportional to PA. Wherefore, &c.

CHAP. XIV.

ON THE MOTION OF PENDULUMS IN RESISTING MEDIUMS.

287. If a pendulum oscillate in a resisting medium, the breadths of the successive vibrations become continually less and less. Nevertheless, provided that the initial breadth be small, and the resistance be also small, these vibrations, although unequal, continue, to sense at least, isochronal with one another, and nearly isochronal with those of a pendulum, of equal length, which oscillates freely in vacuo. The following propositions will fully assure us of this.

288. ***Proposition I.*** If a pendulum oscillate, in a resisting medium, through the cycloidal arc *PAQ* (Fig. 30.) beginning its motion from the point *P*, to find the velocity at any point of the arc.

Here, besides gravity, we must take into account the resistance of the medium, expressed (226.) by gk^2u^2; which force acts throughout the small arc ds, in a direction contrary to that of the actual motion. Resolving this force into two, in the directions of x and y, we shall have the two components

$$gk^2u^2 \cdot \frac{dx}{ds},\ gk^2u^2 \cdot \frac{dy}{ds}.$$

Hence $P = -g + gk^2u^2 \cdot \frac{dx}{ds}$; $Q = gk^2u^2 \cdot \frac{dy}{ds}$.

Therefore (233.) $udu = Pdx + Qdy = -gdx + gk^2u^2ds$. And putting, from the nature of the cycloid, $adx = sds$, and making $2gk^2 = h$, we have

$$2audu - ahu^2ds = -2gsds.$$

This equation is rendered integrable, by multiplying it by e^{-hx}. Integrating it, therefore, and calling c the velocity at the lowest point A, if the constant quantity be determined so that $u = c$ when $s = 0$, there results the equation

$$(M)\ u^2 = e^{hs}c^2 + \frac{2g}{ah^2}\cdot(1 + hs - e^{hs}),$$

which shews the progress of the velocity in the arc of descent PA. Making s negative, we have, similarly, for the arc of ascent AQ, the equation

$$(N)\ e^{-hs}c^2 + \frac{2g}{ah^2}\cdot(1 - hs - e^{-hs}).$$

289. *Coroll.* 1. Let $PA = E$, $AQ = F$: it is manifest that if, in the equation (M) we make $s = E$, then $u = 0$, and that if in the equation (N) we put $s = F$, then also $u = 0$. Hence we may express, in two ways, the velocity C, corresponding to the lowest point.

The equation (M) gives

$$\frac{ah^2c^2}{2g} = -e^{-hE}(1 + hE - e^{hE})$$
$$= \tfrac{1}{2}h^2E^2 - \tfrac{1}{3}h^3E^3 + \tfrac{1}{8}h^4E^4 - \&c.$$

Likewise, the equation (N) gives

$$\frac{ah^2c^2}{2g} = -e^{hF}(1 - hF - e^{hF})$$
$$= \tfrac{1}{2}h^2F^2 + \tfrac{1}{3}h^3F^3 + \tfrac{1}{8}h^4F^4 + \&c.$$

290. *Coroll.* 2. Equating these two expressions, we shall discover the relation between the two arcs E and F; whence, if the first be given, the second will be known. We have this relation from the equation

$$e^{-hE}(1 + hE) = e^{hF}(1 - hF):$$

where, if we make $F = E - mE^2 + nE^3 - pE^4 + \&c.$ and, expanding the whole in series, if we determine the coefficients m, n, p, &c., we shall find

$$F = E - \frac{2}{3}hE^2 + \frac{4}{9}h^2E^3 - \&c.$$

If the arc E be small, and if the resistance be also small, a few terms of this series will suffice. We shall, therefore, at present, take only the three first, neglecting the powers of h that are higher than the square.

291. *Coroll.* 3. The arc of ascent, in the first oscillation, being known, those of the following oscillations are readily found. In the second oscillation, that is, in the return of the pendulum, the arc of descent will be F, and therefore that of the ascent will be

$$F - \frac{2}{3} h F^2 + \frac{4}{9} h^2 F^3,$$

$$\text{or } E - \frac{4}{3} h E^2 + \frac{16}{9} h^2 E^3.$$

Similarly, in the third oscillation the arc of ascent will be found

$$E - \frac{6}{3} h E^2 + \frac{36}{9} h^2 E^3;$$

and in the n^{th} ascent it will be

$$E - \frac{2}{3} n h E^2 + \frac{4}{9} n^2 h^2 E^3.$$

Thus, after n oscillations, the difference between the first descent and the last ascent, will be

$$\frac{2}{3} n h E^2 - \frac{4}{9} n^2 h^2 E^3.$$

Hence it may be seen, how, and by what degrees, the course of the pendulum is continually shortened: and it is remarkable that this shortening does not at all depend upon the length of the pendulum.

292. *Coroll.* 4. It is also worthy of observation that, in this case, the greatest velocity of the oscillating heavy body is no longer at the lowest point A, but at another point B. This point may be found, according to the known rules, by putting the differential of u^2 (288.) equal to nothing.

Let $AB = S$, and the greatest velocity in $B = C$. The equation of the maximum, will be

$$(L) \quad e^{-hS} = 1 - \frac{a h^2 c^2}{2g}.$$

Substitute for $\frac{ah^2c^2}{2g}$ its value (289.) and expanding the expression into series, as was done above (290.), we have

$$S = \tfrac{1}{2}hE^2 - \tfrac{1}{3}h^2E^3 + \tfrac{1}{4}h^3E^4 - \&c.$$

293. *Coroll.* 5. Next, putting, in the equation (*M*), for e^{-hs} its value found from the equation (*L*), we shall have

$$\frac{ah^2C^2}{2g} = hS = \tfrac{1}{2}h^2E^2 - \tfrac{1}{3}h^3E^3 + \tfrac{1}{4}h^4E^4 - \&c.$$

Thus we shall know what is the greatest velocity, and in what point it is acquired.

294. *Proposition II.* The same things being supposed, as in the preceding proposition, to find the time of the oscillation through the arc *PAQ*.

$$\text{Let } PB = q\text{; then } s = q + S = q + \frac{ahC^2}{2g}.$$

Now, if in the equation (*M*) this value be put for s, and also for c^2 its value, deduced from the equation (*L*), then, making $C^2 - u^2 = z$, we have

$$\frac{ah^2z}{2g} = e^{hq} - 1 - hq;$$

whence, developing according to the usual method, (290.)

$$q = \frac{\sqrt{az}}{\sqrt{g}} - \frac{ahz}{6g} + \frac{ah^2z\sqrt{az}}{36g\sqrt{g}} - \&c.$$

$$\text{therefore } dt = -\frac{ds}{u} = \frac{-dq}{\surd(C^2 - z)}$$

$$= -\frac{dz\sqrt{a}}{2\sqrt{g}\sqrt{C^2z - z^2}} + \frac{ahdz}{6g\sqrt{C^2 - z}}$$

$$- \frac{ah^2zdz\sqrt{a}}{24g\sqrt{g}.\sqrt{C^2z - z^2}} + \&c.$$

Integrating, from $u = 0$, $z = C^2$, to $u = C$, or $z = 0$, there results

$$t = \frac{\pi\sqrt{a}}{2\sqrt{g}} - \frac{ahC}{3g} + \frac{\pi h^2 C^2 a\sqrt{a}}{48g\sqrt{g}} - \&c.$$

which will be the time of descent through *PB*.

In like manner will be found the time through *BQ*, by repeating the calculation, upon the supposition that q and dq are negative, and integrating from $z = 0$ to $z = C^2$. This time will be

$$\frac{\pi\sqrt{a}}{2\sqrt{g}} + \frac{ahC}{3g} + \frac{\pi h^2 C^2.a\sqrt{a}}{48g\sqrt{g}} + \&c.$$

And, summing the two times, we shall have the time of the whole course through *PBQ* expressed by

$$\frac{\pi\sqrt{a}}{\sqrt{g}}\cdot\left(1 + \frac{ah^2C^2}{24g} + \&c.\right)$$

295. *Coroll.* 1. If instead of $\frac{ah^2C^2}{2g}$ be put its value (293.) the time of an oscillation will be

$$\frac{\pi\sqrt{a}}{\sqrt{g}}\cdot\left(1 + \frac{1}{24}h^2E^2 - \frac{1}{36}h^3E^3 + \&c.\right).$$

In vacuo, the time of an oscillation would (282.) have been $\frac{\pi\sqrt{a}}{\sqrt{g}}$. Wherefore, the resistance of the medium somewhat retards the oscillation. It is evident, however, that, if h and E be small fractions, this retardation of the pendulum oscillating in a medium, above that of a pendulum oscillating in vacuo, becomes almost insensible; and is the more so, in proportion as the oscillations, being shortened, go on diminishing the arc of descent.

269. *Coroll.* 2. A fortiori, then, is the difference between the time of one oscillation and another, insensible. For in the second oscillation the time is

$$\frac{\pi\sqrt{a}}{\sqrt{g}}\left(1 + \frac{1}{24}h^2F^2 - \frac{1}{36}h^3F^3 + \&c.\right),$$

$$\text{or } \frac{\pi\sqrt{a}}{\sqrt{g}}\left(1 + \frac{1}{24}h^2E^2 - \frac{1}{12}h^3E^3 + \&c.\right);$$

so that the time of the second oscillation is shorter than that of the first, by the time $\frac{\pi\sqrt{a}}{\sqrt{g}} \cdot \frac{h^3 E^3}{18} -$, &c.; a difference, according to our hypothesis, that is inconsiderable.

Wherefore, although, in mediums resisting according to the square of the velocity, the cycloid is not rigorously a tantochronal curve, and the oscillations go on quicker, in proportion as they become shorter, yet, when the arcs are small and the resistance is small, this difference may be neglected.

297. Scholium. All, that has been said of cycloidal pendulums, may be applied to circular pendulums, upon the supposition laid down, that the vibration of the pendulum is confined to arcs of small breadth. Therefore, in these pendulums, also, the air's resistance goes on sensibly shortening the lengths of the oscillations, without, however, sensibly disturbing their isochronism.

ON MOTION.

SECTION II.

ON THE MOTION OF SYSTEMS OF INVARIABLE FORM.

CHAP. XV.

ON THE MOTION OF SYSTEMS IN GENERAL.

298. HAVING seen the laws to which the motion of a point, or elementary particle of matter, is subjected, we are now to consider a system of points, so connected with one another, that, in consequence of this connexion, they cannot follow those movements, which the forces applied separately impress upon them, but are compelled to move in another manner.

299. *PROPOSITION.* If there be a system of points which, by the forces applied, ought to receive the several velocities A, B, C, &c., but, in consequence of their mutual connexion, instead of them, receive the velocities a, b, c, &c.: and if the velocity A be resolved into the actual velocity a and into any other velocity α; similarly, if the velocity B be resolved into the two b, β; and the velocity C into the two c, γ; and so on: then, the forces corresponding to the velocities α, β, γ, &c., are in equilibrium with one another.

For, since the velocities α, β, γ, &c. have no effect, in the system, the forces corresponding to them must needs counterbalance and mutually destroy each other.

300. *Coroll.* Hence, there is an equilibrium between the *impressed* forces, or those corresponding to the velocities A, B, C, &c., and the *actual* forces, or those corresponding to

a, b, c, &c., provided that these last be supposed to act in contrary directions.

For, if A be the resultant of the forces a, α, reciprocally, α will be the resultant of the forces A, $-a$: thus, also, β will be the resultant of B, $-b$; and so on. But (299.) the forces α, β, γ, &c. are in equilibrium. Wherefore, the forces A, B, C, &c., will be in equilibrium with the forces $-a$, $-b$, $-c$, &c.

301. Scholium. This is the principle, which D'Alembert, with such great success, introduced into Dynamics; deducing from it a general method for the solution of very abstruse problems. The method consists in substituting, for the forces impressed on the system, two sets of equivalent forces. Those of the first set tend to impress such motions as each point may freely follow, without causing obstruction to the rest, or suffering obstruction from them: those of the second set, on the contrary, are such as are all in equilibrium with one another. The first are fully effective, and determine the actual motion of the system: the second wholly counteract each other, and determine the pressure which each point of the system suffers, from the mutual action of the parts.

CHAP. XVI.

ON THE MOMENTUM OF INERTIA.

302. The *Momentum of Inertia* of a system in respect of an axis, is the sum of the products which arise from multiplying each element of the system by the square of its distance from the axis.

303. *Coroll.* The momentum of inertia is always a positive quantity, and always increases, when the mass of the system increases.

304. ***Proposition I.*** If S be the momentum of inertia of a system, in respect of an axis which passes through its centre of gravity, S' the momentum of inertia of the same system, in respect of an axis parallel to the first and distant from it by the space k, and M the mass of the system, then

$$S' = S + M.k^2.$$

Let GG' (Fig. 31.) be the axis drawn through the centre of gravity of the system, CC' the other axis parallel to GG'. Let there be at M an element of the system, and call the mass of this element dM. Through M suppose a plane to be drawn perpendicular to the axes GG', CC', and cutting them in C and G'; let there be drawn in it the straight lines MG', MC; let C, G' be joined, and draw to CG' from M the perpendicular MP. Then

$$S = \Sigma.dM.MG'^2; \;\; S' = \Sigma.dM.MC^2.$$

But $MC^2 = MP^2 + CP^2$; and $MP^2 = MG'^2 - G'P^2$; and $CP^2 = (G'P + CG')^2$.

Whence

$$\Sigma.dM.MC^2 = \Sigma.dM.MG'^2 + \Sigma.2dM.CG'.G'P + \Sigma.dM.CG'^2:$$

or, $S' = S' + 2k.\Sigma.dM.G'P + Mk^2$.

Now the product $dM.G'P$ expresses (54.) the momentum of the element dM referred to a plane drawn through $G'G$ perpendicularly to the straight line CG'. Wherefore (36.) $\Sigma . dM . GP = 0$. There remains then $S' = S + Mk^2$.

305. *Coroll.* 1. The momentum of inertia of a system, with respect to any axis, being known, it is easy to find its momentum of inertia with respect to an axis parallel to the first.

306. *Coroll.* 2. Of all the parallel axes, to which a system may be referred, that which passes through the centre of gravity, gives the least momentum of inertia.

307. *Proposition II.* To find the momentum of inertia of a system in respect of a given axis.

The general solution of this problem plainly consists in expressing analytically, by means of the co-ordinates x, y, z, the product of an element multiplied by the square of its distance from the axis, and in afterwards integrating for the whole extent of the system. And instead of the given axis, another parallel to it may be taken, whenever it serves to facilitate the calculation; for, by the preceding proposition, the momentum found may easily be transferred from one axis to the other.

The momentum of inertia of lines and geometrical figures is sometimes investigated, upon the supposition that their elements are endued with a mass proportional to their extension. In the following corollaries we shall assign the momentum of inertia of some of the most simple, thus illustrating the method, above described, by various examples.

308. *Coroll.* 1. To find the momentum of inertia of a straight line of the length a, in respect of an axis drawn perpendicular to the line at its extremity.

The element is dx; the distance from the axis is x. Wherefore, the momentum $= \int x^2 dx = \frac{x^3}{3}$; and completing the integral by putting $x = a$, we have $\frac{a^3}{3}$ for the expression sought.

In this, and in the following examples, we shall put M for the mass of the system. Here $M = a$, and the momentum of inertia is also expressed by $\frac{1}{3} Ma^2$.

If the axis had been drawn through the centre of gravity of the line, that is, through its middle point, the momentum of inertia would have been $\frac{1}{12}a^3$, or $\frac{1}{12}Ma^2$.

309. *Coroll.* 2. To find the momentum of inertia of the periphery of a circle, the radius of which is a, in respect of an axis drawn through the centre, and perpendicular to the plane of the circle.

The element being ds, and its distance from the axis $= a$, the momentum will be $= \int a^2 ds = a^2 s$; and completing the integral by putting $s = 2\pi a$, the momentum $= 2\pi a^3 = Ma^2$.

310. *Coroll.* 3. To find the momentum of inertia of the periphery of a circle, in respect of a diameter of the circle.

The element is $ds = \dfrac{a\,.\,dx}{y}$; its distance from the axis is y. Hence the whole momentum $= a\int y\,dx$. But the whole integral $\int y\,dx$ is the area of the circle itself $= \pi a^2$. Wherefore, the momentum sought is $= \pi a^3 = \frac{1}{2} M a^2$.

311. *Coroll.* 4. To find the momentum of inertia of a circle, in respect of an axis drawn through its centre, and perpendicular to its plane.

Let C (Fig. 32.) be the centre of the circle, and CA the axis perpendicular to its plane; let the radius $CR = a$; in CR take a part $CM = z$, and suppose the circle MM' to be described with the radius $CM = z$, and very near it the circle mm', with the radius $Cm = z + dz$. The area of the zone, comprised between these two circles, will be $2\pi z\,dz$, and its momentum of inertia (309.) will be $2\pi z^3\,dz$. Hence, integrating, we have $\frac{1}{2}\pi z^4$; and completing the integral, by putting $z = a$, the momentum of inertia of the circle is found to be $= \frac{1}{2}\pi a^4 = \frac{1}{2} Ma^2$.

312. *Coroll.* 5. To find the momentum of the same circle in respect of a diameter

The area of the elmentary zone, being, as before, $2\pi z\,dz$, its momentum with respect to the diameter (310.) is $\pi z^3 dz$. Hence, integrating, and afterwards putting $z = a$, the momentum sought $= \frac{1}{4}\pi a^4 = \frac{1}{4} M a^3$.

If it be required to find the momentum of the same circle, with respect to an axis parallel to the diameter, and distant from it by the space k, this will be (304.)

$$\tfrac{1}{4}\pi a^4 + \pi a^2 k^2 = \tfrac{1}{4} M a^2 + M k^2.$$

113. *Coroll.* 6. To find the momentum of inertia of a rectangular parallelepiped, the sides of which are a, b, c, taking for an axis the side c. If the sides a, b, c, be taken for the axes of x, y, z, the element of the solid will be $dx \,.\, dy \,.\, dz$, and the square of its distance from the axis c will be $x^2 + y^2$. The momentum of inertia will, therefore, be $\int (x^2 + y^2)\, dx\, dy\, dz$: and integrating, successively, in relation to each of the three variable quantities x, y, z, there will result

$$\tfrac{1}{3} x^3 yz + \tfrac{1}{3} x y^3 z\,;$$

then, completing the integral by putting $x = a$, $y = b$, $z = c$, we shall have $\frac{1}{3} abc\,(a^2 + b^2)$, or $\frac{1}{3} M(a^2 + b^2)$.

But if the axis pass through the centre of gravity and be still parallel to the side c, the momentum will be $\frac{1}{12} M(a^2 + b^2)$.

314. *Coroll.* 7. To find the momentum of inertia of a solid, generated by the revolution of a curve, in respect of the axis of rotation.

Let RAR' (Fig. 32.) be the curve, which, by revolving about the axis AC, generates the solid; let $AC = x$, $CR = y$, and take for the element of the solid the part intercepted between the circle RR', corresponding to the abscissa x, and the next circle corresponding to the abscissa $x + dx$. The momentum of inertia of this part is (311.) $\frac{1}{2}\pi y^4 dx$; whence that of the solid will be $= \frac{1}{2}\pi \int y^4 dx$.

315. *Coroll.* 8. Applying this formula to a cylinder, to a cone, and to a sphere, we have the following values of the momentum of inertia:

1. For a cylinder, a being the radius of its base, and b its length,

$$S = \tfrac{1}{2}\pi a^4 b = \tfrac{1}{2} M a^2.$$

2. For a cone, a being the radius of its base, and b its altitude,

$$S = \tfrac{1}{10}\,\pi a^4 b = \tfrac{3}{10}\,M a^2.$$

3. For a segment of a sphere, a being the sphere's radius, and x the sagitta of the segment,

$$S = \pi x^3 \left(\tfrac{2}{3}a^2 - \tfrac{1}{2}a x + \tfrac{1}{10}x^2\right).$$

4. Hence for the hemisphere,

$$S = \tfrac{4}{15}\,\pi a^5 = \tfrac{2}{5}\;M a^2:$$

and for the whole globe,

$$S = \tfrac{8}{15}\,\pi a^5 = \tfrac{2}{5}\;M\,.\,a^2.$$

316. *Coroll.* 9. To find the momentum of inertia of the solid generated by the revolution of the curve RAR', referred not to the axis of rotation CA, but to an axis drawn perpendicular to CA, through its extremity A.

The elementary portion of the solid intercepted between the abscissas x and $x + dx$, may be considered as a circle, of the radius y, distant from the axis by the interval x; and its mass $M = \pi y^2 dx$. Hence (312.) the element of momentum will be $\frac{1}{4}\,\pi y^4 dx + \pi x^2 y^2 dx$; and the whole momentum will be

$$\tfrac{1}{4}\,\pi\!\int y^4 dx + \pi\!\int x^2 y^2 dx.$$

If the axis pass, not through the vertex A, but through the centre of gravity of the solid, then calling X the abscissa of the centre of gravity, and M the mass of the solid, we shall have the momentum of inertia (304.)

$$\tfrac{1}{4}\,\pi\!\int y^4 dx + \pi\!\int x^2 y^2 dx - MX^2.$$

317. *Coroll.* 10. Applying this last formula to the cylinder, to the cone, and to the sphere, there result the following values for the momentum of inertia of these solids, referred to an axis drawn through their centre of gravity, perpendicularly to the axis of rotation; retaining the notation of Art. 315:

1. For the cylinder . . . . . $S = \frac{1}{4} M a^2 + \frac{1}{12} M b^2$.

2. For the cone . . . . $S = \frac{3}{20} M a^2 + \frac{3}{80} M b^2$.

3. For the hemisphere. . . . $S = \frac{83}{320} M a^2$.

4. For the ellipsoid, a being the semi-axis of rotation, and b the semi-conjugate axis, the momentum of inertia, in respect of a, is $S = \frac{2}{5} M b^2$; and, in respect of b, it is $S = \frac{1}{5} M (a^2 + b^2)$.

CHAP. XVII.

ON PRINCIPAL AXES.

318. A SYSTEM being referred to three rectangular axes OX, OY, OZ, (Fig. 33.) of the co-ordinates x, y, z, if we call dM the element of the mass, and put

$$\int x^2 dM = A;\ \int y^2 dM = B;\ \int z^2 dM = C,$$
$$\int xy\, dM = D;\ \int xz\, dM = E;\ \int yz\, dM = F;$$

the momentum of inertia of the system, in respect of the axis OX, will be $S(y^2+z^2)\,dM = B + C$; in like manner, the momentums of inertia, for the other two axes OY, OZ, will be $A + C$, $A + B$.

319. *PROPOSITION I.* The momentums of inertia being given, for the three axes OX, OY, OZ, (Fig. 33.) to find the momentum of inertia for any other axis OG, drawn through the origin, O, of x, y, z.

Let OF be the projection of OG on the plane XOY; and let the angles $XOF = p$, $FOG = q$. Let the element dM, at the point M, be determined by the co-ordinates $OP = x$, $PQ = y$, $QM = z$. In the plane XOY let QR be drawn perpendicular to OF, and let the co-ordinates be changed, by making $OR = x'$, $RQ = y'$, $QM = z'$. Then

$$x' = OQ \cos.(POQ - p) = x \cos. p + y \sin. p;$$
$$y' = OQ \sin.(POQ - p) = y \cos. p - x \sin. p,$$
$$z' = z.$$

Complete the rectangle $RQMT$, and draw TS perpendicular to OG; complete also the rectangle $MTSK$, and, again changing the co-ordinates, make $OS = x''$, $SK = y''$, $KM = z''$. Wherefore

$$x'' = OT \cos.(ROT - q) = x' \cos. q + z' \sin. q,$$
$$y'' = y',$$
$$z'' = OT \sin.(ROT - q) = z' \cos. q - x' \sin. q.$$

Substituting here the values of x', y', z', and thence deducing the value of $y''^2 + z''^2$, we shall have the momentum of inertia for the axis OG; which, if it be called Σ, will be

$$\Sigma = \int (y''^2 + z''^2)\, dM =$$

$$A\,(\sin^2. p + \cos.^2 p . \sin.^2 q) + B\,(\cos.^2 p + \sin.^2 p \sin.^2 q)$$

$$+ C \cos.^2 q - 2\,D \sin. p . \cos. p . \cos.^2 q$$

$$- 2\,E \cos. p . \sin. q . \cos. q$$

$$- 2\,F \sin. p . \sin. q . \cos. q .$$

320. *Coroll.* With the same data, may be calculated the sums $\int x''y''dM$, $\int x''z''dM$; and comparing the values of these sums with that of Σ already found, we shall have

$$\int x''y''dM = -\frac{1}{2\cos. q}\left(\frac{d\Sigma}{dp}\right).$$

$$\int x''z''dM = -\tfrac{1}{2}\left(\frac{d\Sigma}{dq}\right).$$

321. *Proposition II.* Amongst all the axes drawn through the origin O (Fig. 33.), to find the axis OG, to which if the momentum of inertia of the system be referred, it shall be a maximum or a minimum.

The problem is solved from the two equations

$$\left(\frac{d\Sigma}{dp}\right) = 0; \quad \left(\frac{d\Sigma}{dq}\right) = 0.$$

Differentiating, therefore, the value of Σ (319.) separately, with respect to the two variable quantities p, q, and putting both the differentials equal to nothing, we shall have two equations to determine the two angles p and q, which fix the position of the axis required.

When q is eliminated, there will remain, to determine the angle p, a complicated cubic equation of this form,

$$A' \tan.^3 p + B' \tan.^2 p + C' \tan. p + D' = 0.$$

322. *Coroll.* 1. This equation being a cubic, must have one real root; it will have two: for the momentum of inertia for every axis being a positive quantity, if there be one axis which gives the momentum a maximum, there must necessarily be another which gives it a minimum, and vice versâ. But if two

roots be real, the third must also be real. The equation, then, will have all its three roots real, and therefore, generally speaking, will indicate three axes.

323. *Coroll.* 2. But the condition of a maximum or a minimum cannot belong to all these three axes; so that if one of them absolutely give the greatest momentum, and another the least, the third cannot give either a maximum or a minimum. It is, indeed, well known, that, from the differential of a function being equal to nothing, it cannot always be concluded that the function is a maximum or minimum.

There will, however, belong to all the three axes, this property, that the differential of their momentum of inertia is equal to nothing: that is to say, that a very small change in the position of the axis in any direction, will not at all change the value of the momentum of inertia.

324. *Coroll.* 3. These axes are called *Principal Axes* of the system. A principal axis is, therefore, that for which the differential of the momentum of inertia is equal to nothing. And it appears, that, through each point of the system, there may be drawn three axes endued with this property.

325. *Coroll.* 4. The equations $\left(\frac{d\Sigma}{dp}\right) = 0$, $\left(\frac{d\Sigma}{dq}\right) = 0$, by which the principal axes are determined, give also the two others (320.) $\int x''y''\,dM = 0$, $\int x''z''\,dM = 0$. Wherefore, every principal axis has this property, that, if the abscissas x be taken in it, $\int xy\,dM = 0$, $\int xz\,dM = 0$: and vice versâ.

326. *Coroll.* 5. If the system be symmetrical about the axis OG, OG will be a principal axis.

For x being taken in the axis OG, for every element dM determined by the co-ordinates x, y, there will be another equal to it, determined by the co-ordinates x, $-y$. Whence it is manifest, that $\int xy\,dM = 0$; and in the same manner it will be found that $\int xz\,dM = 0$. Wherefore, &c. (325.).

327. SCHOLIUM. To find the principal axes by means of a cubic equation, would be a very laborious operation. But, in symmetrical systems, the preceding corollary indicates one at

least of these axes. And when one of them is known, the other two are easily discovered by means of the following proposition.

328. *Proposition III.* One principal axis of a system being given, to find the other two.

Let OX be the principal axis which is given. Taking x in this line, we may make (319, 325.) $D = E = 0$. In order, therefore, to find another axis endued with the same property, in the value of Σ (319.) which expresses the momentum of inertia of any axis whatever, put $D = E = 0$; and there result the two equations $\left(\frac{d\Sigma}{dp}\right) = 0$, $\left(\frac{d\Sigma}{dq}\right) = 0$, by which the angles p and q are determined.

Now, since $\sin.^2 q = 1 - \cos.^2 q$, the first equation will be found to be

$2(A - B) \sin. p \cos. p \cos.^2 q - 2F \cos. p \sin. q \cos. q = 0$;
whence, $\cos. p = 0$, and therefore $\sin. p = 1$.

Again, since $2 \sin. q \cos. q = \sin. 2q$; $\cos.^2 q - \sin.^2 q = \cos. 2q$, the other equation is found to be

$(A \cos.^2 p + B \sin.^2 p - C) \sin. 2q - 2F \sin. p \cos. 2q = 0$;
whence, since $\cos. p = 0$, and $\sin. p = 1$,

$$\tan. 2q = \frac{2F}{B - C}.$$

From the first equation, therefore, we have the angle $p = 90°$. The second equation, if $2i$ be the angle which has $\frac{2F}{B - C}$ for its tangent, gives the two values $q = i$, $q = i + 90°$, to determine the angle q.

Hence the other two principal axes, which go along with OX, will be 1st, Oy drawn in the plane YOZ, at the angle $YOy = i$; 2dly, Oz, which, on the same plane, makes the angle $YOz = 90°$.

329. *Coroll.* 1. Wherefore, the three principal axes are at right angles to each other.

330. *Coroll.* 2. If we take, on the principal axes, the coordinates x, y, z, we shall have

$$\int xy\,dM = 0; \quad \int xz\,dM = 0; \quad \int yz\,dM = 0:$$

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And if, further, we place the origin of x, y, z, in the centre of gravity, we shall also have

$$\int x\,dM = 0;\ \int y\,dM = 0;\ \int z\,dM = 0.$$

331. *Proposition IV.* The momentums of inertia for the three principal axes being given, to find the momentum of inertia for any axis whatever drawn through their origin.

Let OX, OY, OZ, (Fig. 33.) be the three principal axes, and let A', B', C', be their given momentums of inertia: let any other axis OG make with the principal axes the angles $GOX = f$, $GOY = g$, $GOZ = h$. The momentum of inertia for the axis OG will be

$$\Sigma = A'.\cos.^2 f + B'.\cos.^2 g + C'.\cos.^2 h.$$

Taking the co-ordinates in the principal axes, we may (330.) in the value of Σ (319.) put $D = E = F = 0$. And, instead of A, B, C, putting their values given from the equations (318.) $B + C = A'$, $A + C = B'$, $A + B = C'$, we shall have

$$\Sigma = A'\cos.^2 p \cos.^2 q + B'\sin.^2 p \cos.^2 q + C'\sin.^2 q.$$

$$\text{But } \cos. p.\cos. q = \cos. GOX = \cos. f;$$

$$\sin. p.\cos. q = \cos. GOY = \cos. g;$$

$$\sin. q = \cos. GOZ = \cos. h.$$

$$\text{Wherefore } \Sigma = A'\cos.^2 f + B'\cos.^2 g + C'\cos.^2 h.$$

332. *Coroll.* 1. Since the three principal axes are at right angles to each other, $\cos.^2 f + \cos.^2 g + \cos.^2 h = 1$. Whence two of the three angles f, g, h, are sufficient to determine the position of the axis OG.

333. *Coroll.* 2. Wherefore, the value of Σ may also be expressed in this manner:

$$\Sigma = A' - (A' - B')\cos.^2 g - (A' - C')\cos.^2 h;$$

$$\text{or } \Sigma = C' + (A' - C')\cos.^2 f + (A' - B')\cos.^2 g.$$

Let $A' > B' > C'$; then it is manifest that $A' > \Sigma > C'$: which confirms the proposition that the momentum of inertia of any axis is comprised between the greatest momentum A', and the least C', belonging to two principal axes.

334. *Coroll.* 3. It may happen that, amongst the momentums of the principal axes, there are two that are equal to one another.

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Let these be the momentums of the axes OX, OY, so that $A' = B'$. Then

$$\Sigma = A'\,(1 - \cos.^2 h) + C' \cos.^2 h;$$

which value $= A'$, when $h = 90°$. Wherefore, all the axes, drawn in the plane XOY, will have equal momentums of inertia.

335. *Coroll.* 4. It may also happen that the momentums of inertia of the principal axes are all three equal to each other; whence $A' = B' = C'$. Then, $\Sigma = A'$, and all the axes have the same momentum of inertia.

336. *Coroll.* 5. Hence it appears, that the principal axes, which can be drawn through any point whatever of a system, are always either three in number, or infinite in number.

337. SCHOLIUM. From what has been said the principal axes of the most simple figures are easily found.

In the rectangular parallelepiped, the principal axes, drawn through the centre of gravity, are three straight lines parallel to the sides.

In the sphere, every diameter may be taken as a principal axis.

In the circle, as well for the periphery as for the area, one of the principal axes is a straight line, drawn through the centre, perpendicular to the plane of the circle; the other two are any two diameters at right angles to each other: for every diameter, being similarly posited with respect to the circle, has an equal momentum of inertia.

In like manner, in solids generated by revolution, the axis of revolution is itself one of the principal axes; and the other two are any two straight lines, perpendicular to the axis, which cut one another at right angles.

CHAP. XVIII.

ON THE MOTION OF A RIGID SYSTEM ABOUT AN IMMOVEABLE AXIS.

338. WHILST a body, or a rigid system, revolves about an immoveable axis, each point of it describes the circumference of a circle, which has its centre in the axis, and which has for radius the distance of that point from the same axis. Hence, the contemporaneous velocities of different points are proportional to their distances from the axis. The velocity of those points, the distance of which from the axis $= 1$, is called the *Angular Velocity* of the system.

If the angular velocity be known, the velocity of each point in the system is known. Let the angular velocity $= \omega$; then, the velocity of a point, or element, at the distance r from the axis of rotation $= r\omega$.

339. *PROPOSITION I.* If a body, moveable about an axis, be urged by a force acting in a plane perpendicular to the axis, the body will begin to revolve with an angular velocity equal to the momentum of the force divided by the momentum of inertia of the body; both momentums being referred to the immoveable axis.

Let F be the force; a the distance of its direction from the axis; S the momentum of inertia of the body in respect of the axis; then the angular velocity will be $\omega = \frac{aF}{S}$.

For, let dM be an element situated at the distance r from the axis of rotation. This element having, in virtue of the force F, received the velocity $r\omega$, and thence the force $r\omega dM$, if it turned round in a contrary direction, and if all the other elements did the same, the aggregate of these forces must (300.) be in equilibrium with the force F. Hence the sum of their momentums of rotation must (108.) be equal to the momentum aF of the force F. But the momentum of the elementary force $r\omega\, dM = r^2\omega\, dM$, and the sum of these momentums $= \omega\int r^2 dM = \omega S$. Wherefore, $\omega S = aF$; and $\omega = \frac{aF}{S}$.

340. *Coroll.* 1. If the force F, after its first impulse, cease to act, the body will go on continually revolving uniformly about the axis, and always (254.) with the angular velocity above determined.

341. *Coroll.* 2. But if the action of the force F is continual, the rotatory motion will be accelerated, and we shall have

$$\frac{d\omega}{dt} = \frac{aF}{S}.$$

For, let $d\omega$ be the increment which the angular velocity receives, in the instant of time dt. The increment of velocity, which the element dM receives, will be $r d\omega$; whence its accelerating force (203.) will be $\frac{r d\omega}{dt}$, and its moving force will be $\frac{r.d\omega.dM}{dt}$. Now all these forces, acting in a contrary direction, must (300.) counterbalance the force F, and, therefore, the sum of their momentums $= a\ F$. Hence $\frac{d\omega}{dt}\int r^2 dM = \frac{S d\omega}{dt} = aF$.

342. Scholium. We have next to investigate the stress sustained by the axis upon which the body revolves. And, first, we shall consider the stress sustained by it, at the first movement of the body, in consequence of the impulse communicated by the force F; then, the stress sustained in the continuation of the motion, by reason of the motion itself, and independently of the force which has produced it.

343. *Proposition II.* To determine the stress which the axis sustains, in consequence of the impulse of the force F.

Let the axis of rotation be OZ (Fig. 15.) fixt in the two pivots M, N. Let the system be referred to three rectangular axes; take OZ for the axis of z, and the force F lying in the plane of x, y, let the axis of x be perpendicular to the direction of that force. Let $OM = \zeta$, $ON = \zeta'$, be the ordinates which determine the places of the two pivots. Take any element dM, to which belong the co-ordinates x y, z; its distance from the axis OZ will be $\sqrt{(x^2 + y^2)}$; hence its velocity $= \omega \sqrt{(x^2 + y^2)}$, and its force $= \omega d M \sqrt{(x^2 + y^2)}$. This force is in the direction of the tangent of the circle, which the point describes in its revolution, produced on that side towards which the rotation takes place. Hence, resolving it into two, the one parallel to x, the other to y, the first will be found $= - \omega y \, d M$, the second $= \omega x . d M$. Wherefore, the sum, or resultant, of the first will be $= - \omega \int y \, d M$, and that of the others $= \omega \int x \, d M$. These forces, acting in a contrary direction, must (300.) counterbalance the force F, which by the hypothesis, acts in the direction of y. Thus, in the system, there is an equilibrium between the force $\omega \int y \, d M$ acting in the direction of x, and the force $F - \omega \int x \, d M$ acting in the direction of y. Calling, therefore, p, q, the pressures sustained by the fulcrum M in the direction of x and of y, and p', q', the pressures similarly sustained by the fulcrum N, there results (129.)

$$p + p' = \omega \int y \, d M; \quad q + q' = F - \omega \int x \, d M;$$
$$p\zeta + p'\zeta' = \omega \int y z \, d M; \quad q\zeta = q'\zeta = - \omega \int x z \, d M;$$

whence may be determined the stresses sought.

344. Scholium 1. If the force lies in a plane oblique to the axis OZ, it becomes necessary to resolve it into two, the one parallel to OZ, the other, F, situated in a plane perpendicular to OZ. The first of these forces will produce a further stress on the pivots M, N, which must be determined as in Art. 131, and must be added to the stress arising from F.

345. Scholium 2. The stress, which the impulse of the force produces in the first movement of the body, is soon counteracted by the resistance of the pivots M, N. This is not the case with the pressures which arise from the rotation of the body; these last as long as the rotation itself continues. We shall determine their values in the following proposition.

346. *Proposition III.* To determine the stress sustained by the axis during the rotation of the body about it.

Each element dM describing the circumference of a circle, which has for radius $\sqrt{(x^2 + y^2)}$, its centrifugal force will be expressed (260.) by $\omega^2 dM \sqrt{(x^2 + y^2)}$; and this force acts in the direction of the radius itself produced. Resolving it into two forces parallel to x and y, these are $\omega^2 x dM$, $\omega^2 y dM$. Now the aggregate of these forces is sustained and counterbalanced by the resistance of the pivots. Hence, calling p, q, the pressures sustained by the pivot M, in the directions of x, y, and p', q', those similarly sustained by the pivot N, we shall have (129.)

$$p + p' = \omega^2 \int x dM; \quad q + q' = \omega^2 \int y dM;$$

$$p\zeta + p'\zeta' = \omega^2 \int xz dM; \quad q\zeta + q'\zeta' = \omega^2 \int yz dM;$$

whence the required pressures may be determined.

347. *Coroll.* 1. If the axis OZ be one of the three principal axes which pass through the centre of gravity of the body, the pivots sustain no pressure whatever.

For, OZ being a principal axis, we have (325.)

$$\int xz dM = \int yz dM = 0;$$

and since the centre of gravity is in the axis OZ, we have also (56.) $\int x dM = \int y dM = 0$: wherefore (346.), &c.

348. *Coroll.* 2. If OZ is a principal axis, but does not pass through the centre of gravity, then one pivot will suffice to sustain the axis of rotation.

For, suppose the origin of the abscissas to be placed in M. Then, $\zeta = 0$, $\int xz dM = \int yz dM = 0$.

Substituting these values, we find (346.) $p' = 0$, $q' = 0$; wherefore, the pivot N will not sustain any pressure; and the other pivot M will sustain the pressures $p = \omega^2 \int x dM$; $q = \omega^2 \int y dM$.

CHAP. XIX.

ON THE CENTRE OF PERCUSSION.

349. *Proposition I.* If a body revolve about any one of its principal axes, to determine the resultant of the elementary forces.

Let MRD (Fig. 34.) be the section of the body, revolving about the axis CV, made by a plane perpendicular to that axis, and passing through the centre of gravity G. Let M be the mass of the body, S its momentum of inertia with respect to the axis CV, ω the angular velocity, and let the distance $CG = k$. Let CV be taken for the axis of z, let MRD be the plane of x, y, and let CG be the axis of x.

Resolving the force of the element dM according to the directions of x and y, the resultant of all the forces parallel to x will be $= -\omega \int q\,dM$; and the resultant of all the forces parallel to q will be $= \omega \int x\,dM$ (343.). These resultants will both of them fall in the plane MRD; because their momentums (55.), in respect to this plane must be equal to the sum of the momentums of the forces, that is, to $-\omega \int yz\,dM$, and $\omega \int xz\,dM$; and CV being a principal axis, we have (325.) $\int yz\,dM = \int xz\,dM = 0$.

Again, calling X, Y, the co-ordinates of the centre of gravity, we have (50.) $\int x\,dM = MX, \int y\,dM = MY$. But, by the construction, $X = k$, $Y = 0$. Wherefore, the resultant parallel to x is $= 0$; and the other resultant $= Mk\omega$, and is perpendicular to CG.

Now, we shall determine the point Q, in which the direction of this resultant cuts CG, by considering again that the momentum of each of the two resultants, parallel to x and y, with respect to the planes of the co-ordinates must be equal to the sum of the momentums of the components. Hence, for the forces parallel to x, we shall have $\omega\int y^2 dM = 0$; and for those parallel to y, we shall have $Mk\omega.CQ = \omega\int x^2 dM$. Summing these two equations, there results

$$Mk\omega \, . \, CQ = \omega\int(x^2 + y^2)dM = \omega S.$$

Whence $CQ = \dfrac{S}{M.k}$. Thus the required resultant of the elementary force is fully determined both in quantity and position.

350. *Proposition II.* In a body revolving about a principal axis, there is a point, called the *Centre of Percussion*, through which passes the resultant of all the elementary forces, whatever be the position of the revolving body, and whatever be the velocity of rotation.

In the plane MRD (Fig. 34.) perpendicular to the axis CV, and passing through the centre of gravity G of the body, draw CG, and produce it to Q so that $CQ = \dfrac{S}{Mk}$. Then shall Q be the centre of percussion.

For (349.) the resultant of the elementary forces will pass through Q, and $CQ = \dfrac{S}{M.k}$; and since this formula, which determines the place of Q, does not contain either ω, or F, or a, or any element depending upon the positions which the revolving body assumes, it is plain that the resultant will constantly pass through the point Q.

351. *Coroll.* 1. The centre of percussion is always further from the axis of rotation than the centre of gravity.

For, since the momentum of inertia, in respect of an axis which does not pass through the centre of gravity is always (304.) greater than Mk^2, therefore $\dfrac{S}{Mk} > k$, or $CQ > CG$.

352. *Coroll.* 2. If the body is urged to revolve by a force F, acting in the plane MRD, perpendicularly to CG, and which

passes through the centre Q of percussion, the impulse of this force will not produce any stress on the pivots which sustain the axis of rotation.

This follows from the formulas of Art. 343. For since, in this case,

$$\int yz\,dM = \int xz\,dM = \int y\,dM = 0,$$

and that $\int x\,dM = Mk$, we shall have

$$p + p' = 0,\ p\zeta + p'\zeta' = 0,\ q\zeta + q'\zeta' = 0;$$

and $q + q' = F - Mk\omega$. But (339.) $\omega = \frac{aF}{S}$, and here, by the hypothesis, $a = CQ = \frac{S}{Mk}$;

$$\text{whence } \omega = \frac{F}{Mk},\ \text{ or } F - Mk\omega = 0.$$

Wherefore, also, $q + q' = 0$; therefore, &c.

CHAP. XX.

ON THE CENTRE OF OSCILLATION.

353. **A** *Compound Pendulum* is a body, or a system of invariable form, which, left to the action of its own gravity, oscillates about an horizontal axis.

This differs from the simple pendulum, in which the mass is either not at all considered, or is supposed to be all concentrated in a single point.

354. ***Proposition.*** In a compound pendulum there is a point, called the *Centre of Oscillation,* in which if the whole mass of the oscillating body is supposed to be concentrated, the oscillations of this supposititious simple pendulum will be isochronal with those of the compound pendulum.

Let MRD (Fig. 34.) be a section of the pendulum oscillating by its gravity g, about the horizontal axis CV, the section being made by a plane, perpendicular to the straight line CV, drawn through the centre of gravity G. Let the straight line CG make, with the vertical line CT, the angle $GCT = \phi$. Retaining the foregoing denominations, it is plain, that, as the whole weight of the system may be supposed to be collected in its centre of gravity G, the force which urges the system to turn about its axis $= Mg$, and that its momentum with respect to the axis CV will be $Mgk \sin. \phi$; wherefore (341.) the oscillating motion of the compound pendulum will be determined by the equation

$$\frac{d\omega}{dt} = \frac{Mgk \sin. \phi}{S}.$$

Again, produce CG to Q, so that $CQ = \frac{S}{Mk}$; and suppose the whole mass M to be concentrated in Q. Then $k = CQ$, and $S = M.\overline{CQ}^2$; and the oscillatory motion will depend upon the equation $\frac{d\omega}{dt} = \frac{Mg\sin.\phi.CQ}{M.\overline{CQ}^2} = \frac{g\sin.\phi}{CQ}$; or, since

$$CQ = \frac{S}{Mk}, \text{ upon } \frac{d\omega}{dt} = \frac{Mgk\sin.\phi}{S}.$$

But this equation is identical with the preceding equation. Wherefore, the oscillatory motion of the simple pendulum CQ is the same as that of the compound pendulum.

355. *Coroll.* 1. The centre of oscillation is, therefore, found in a manner altogether analogous to that by which the centre of percussion is found. If the body be cut by a plane, perpendicular to the axis, and drawn through the centre of gravity G, and if, in CG produced, there be taken $CQ = \frac{S}{Mk}$, CQ will be the length of the simple pendulum that is isochronal with the compound pendulum.

356. *Coroll.* 2. Hence we have an easy method of finding, mechanically, the momentum of inertia of a body, however irregular it may be.

Make the body oscillate, through very small arcs, about an horizontal axis; and, counting the oscillations made in a given time, the time of half an oscillation is easily found. Thence, by the formula of Art. 284., the length CQ of an isochronal simple pendulum will be known. Lastly, the equation $S = Mk.CQ$ will shew the value of S referred to the axis of rotation.

357. *Coroll.* 3. Let it be required to find the centre of oscillation of geometrical lines and figures, supposing them to be heavy and homogeneous.

Still calling k the distance of the centre of gravity from the axis of rotation, and L the distance of the centre of oscillation from the same axis, we shall have

1. For a straight line of the length a, which oscillates suspended from its upper extremity,

$$L = \frac{2}{3}a = k + \frac{1}{6}a.$$

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2. For a rectangular parallelepiped, having for sides a, b, c, and suspended from an axis which bisects the upper base $b \cdot c$, and is parallel to the side c,

$$L = \frac{1}{6} \cdot \frac{b^2 + 4a^2}{a} = k + \frac{1}{6} \cdot \frac{a^2 + b^2}{a}:$$

which value, if the side b be very small, coincides with that which belongs to a straight line.

3. For a sphere of the radius a,

$$L = k + \frac{2}{5} \cdot \frac{a^2}{k}.$$

4. For a spherical segment, a being the radius, and x the sagitta,

$$L = k + \frac{x}{k} \cdot \frac{\frac{2}{3}a^2 - \frac{1}{2}ax + \frac{1}{10}x^2}{a - \frac{1}{5}x}.$$

The value of L is the same for a line composed of two spherical segments.

358. *Coroll.* 4. The following is an useful rule for finding the centre of oscillation of a system, when the centres of gravity, and the centres of oscillation of its several parts, are known.

Let M', M'', M''', &c., be the masses which form the system; k', k'', k''', &c., the distances of their centres of gravity from the common axis of rotation; l', l'', l''', &c., the distances of their centres of oscillation from that axis: then

$$L = \frac{M'k'l' + M''k''l'' + M'''k'''L''', \text{\&c.}}{M'k' + M''k'' + M'''k''', \text{\&c.}}.$$

For, since $L = \frac{S}{Mk}$, it is evident that the momentum of inertia of the first mass will be $M'kl'$, that of the second $M''k''l''$, and so on; and the momentum of inertia of the whole system, being the sum of the momentums of inertia of its parts, will be $M'k'l' + M''k''l'' +$ &c. Also, by the property of the centre of gravity (55.) $Mk = M'k' + M''k'' +$ &c. Wherefore, &c.

359. *Coroll.* 5. The centre of oscillation is not one single point of the system.

For, if through Q (Fig. 34.) be drawn the horizontal straight line QV', parallel to the axis CV, it is manifest that each point of QV' oscillates precisely as the point Q. Wherefore, all the points of QV' are centres of oscillation; and hence QV' is called the *Axis of the Centres of Oscillation.*

360. *Coroll.* 6. The axis of the centres of oscillation and the axis of suspension reciprocally correspond with each other; so that if QV' becomes the axis of suspension, CV will become the axis of the centres of oscillation.

For, if QV' become the axis of suspension, then $k = QG$, and the momentum of inertia referred to the new axis QV' will be found by Art. 305. to be

$$= S - M(\overline{CG}^2 - \overline{QG}^2)$$
$$= S - M(CG + QG).(CG - QG)$$
$$= S - M.CQ.CG + M.CQ.QG.$$

But $S = M.CQ.CG$; therefore, the momentum of inertia is reduced to $M.CQ.QG$: whence $L = \dfrac{M.CQ.QG}{M.QG} = CQ$.

Wherefore, &c.

CHAP. XXI.

ON THE INITIAL MOVEMENT OF A RIGID AND FREE SYSTEM ACTED UPON BY ONE GIVEN FORCE.

361. WE shall understand what must happen to a body, or any other rigid system, acted upon by several forces, if we first ascertain that which happens to it when it is subjected to the action of one single force. And here it is proper to distinguish two cases; inasmuch as the direction of the force may pass through the centre of gravity, or not pass through it.

362. *PROPOSITION I.* If the direction of a single force, acting upon a body, pass through the centre of gravity, the body will acquire from it a progressive motion, with a direction parallel to that of the force.

For, since the moving force passes through the centre of gravity, it may be resolved into several equal and parallel forces, applied to each of the equal elements of the system; which will, therefore, move no otherwise than if every one of its elements were acted upon by equal and parallel forces: it will, therefore, have a progressive motion.

363. *Proposition II.* The same supposition being made, as that of the preceding proposition, the velocity of the body will be equal to the moving force divided by the mass of the body.

Let M be the mass, F the impelling force. If the force F be resolved into as many elementary equal and parallel forces as there are equal elements of the mass, it is clear that, as the sum of all those elementary forces must $= F$, and their number $= M$, each of them must $= \frac{F}{M}$. Whence (10.) the velocity $u = \frac{F}{M}$.

364. *Proposition III.* If the direction of a single force, acting upon a body, does not pass through the centre of gravity, the body will acquire simultaneously two motions; the one progressive, as if the force had passed through the centre of gravity; the other rotatory about the centre of gravity, as if that centre had been immoveable.

Let MRD (Fig. 35.) be a section of the body, made by a plane, passing through the centre of gravity G, and through the direction of the impelling force AP. Through G, perpendicular to this plane, let there be drawn the axis GX. Draw, also, GA perpendicular to AP; and, GB being taken equal to GA, suppose the two opposite forces BQ, BS, each equal to the half of AP, to be applied to the point B. Let the force AP be divided into two equal forces AK, KP.

Thus, for the force AP, we may substitute the four equal forces AK BQ, KP, BS. Now the resultant of the two first AK, BQ, passes through G and is parallel to AP, and equal to $AK+BQ$, or to AP. Wherefore, as far as regards the two forces AK, BQ, the body will move exactly as if the force AP passed through the centre G.

The other two forces, KP, BS, manifestly tend to turn the body about the axis GX, in the direction of the force AP, and with the momentum $KP \,.\, GA + BS \,.\, GB$. And, since $GB = GA$, and $KP + BS = AP$, this momentum becomes $AP \,.\, GA$, which is the momentum of the force AP to turn the body about GX. Wherefore, by the two forces KP, BS, the body will revolve about the axis GX, as it would do by the sole force AP, if that axis were immoveable.

365. *Proposition IV.* The same supposition being made, as that of the preceding proposition, the velocity of the progressive motion is equal to the moving force divided by the mass of the body; and the angular velocity of rotation is equal to the momentum of the force divided by the momentum of inertia of the body; both the momentums being referred to the axis of rotation GX.

Let the moving force $= F$; the mass of the body $= M$; its momentum of inertia $= S$; the distance $AG = a$; the

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velocity of the progressive motion $= u$; the angular velocity $= \omega$: then shall $u = \frac{F}{M}$; $\omega = \frac{aF}{S}$.

For the progressive motion is the same (364.) as if the force F passed through the centre of gravity: wherefore (363.), $u = \frac{F}{M}$.

Also (364.) the rotatory motion is the same as if the force F turned the body about the axis GX. Wherefore (339) $\omega = \frac{aF}{S}$.

366. *Coroll.* 1. The velocity of the progressive motion is to the angular velocity of rotation as $\frac{F}{M}$ to $\frac{aF}{S}$, or as S to aM. Hence may be easily solved the following problem.

To find in what point a given body must be impelled, so that acquiring contemporaneously a progressive and a rotatory motion, the velocity of the first may be to that of the second in the given ratio of m to n.

We must have $m : n :: S : aM$; whence $a = \frac{nS}{mM}$. Wherefore the direction of the impelling force must be distant from the centre of gravity by the space $\frac{nS}{mM}$.

367. *Coroll.* 2. In the straight line AG produced (Fig. 35.) there is a point, called the *Centre of Spontaneous Rotation*, which for any given instant is at rest; because, as much as it advances by the progressive motion, it recedes by the rotatory motion.

For the velocity of any point of the body is the resultant of the progressive and rotatory velocities which belong to it. Now, for the points situated in the straight line AGB, the two velocities are conspiring, or else opposite. Let, therefore, a point be taken in AB at the distance r from the axis; its velocity will be $u \pm r\omega$, the sign $+$ belonging to points between A and G, and the sign $-$ to points which lie from G towards B. For these latter, then, the velocity being $u - r\omega$, it will $= 0$, in the point where $r = \frac{u}{\omega}$ or $\frac{S}{aM}$. Hence, taking $GC = \frac{S}{aM}$, C will be the centre of spontaneous rotation.

368. *Coroll.* 3. Hence it follows, that the double motion, with which the body is endued, may, for an instant, be considered as a simple rotatory motion about an axis parallel to the axis GX, and distant from it by the quantity $\frac{u}{\omega}$.

And, vice versâ, if the body have only a simple rotatory motion about any axis, the distance of which from the centre of gravity is expressed by $\frac{u}{\omega}$, this motion may for an instant be considered as compounded of two simultaneous motions, the one progressive with the velocity u, the other rotatory about G with the angular velocity ω.

369. SCHOLIUM. If several forces act upon a body, its instantaneous motion will be compounded of all the progressive and rotatory motions due to each of the impelling forces. The composition of progressive motions is already sufficiently known: it remains for us to explain the composition of rotatory motions, without which it cannot well be understood what will be the initial velocity of a body acted upon, at once, by several different forces; much less can the course of the motion be determined.

CHAP. XXII.

ON THE COMPOSITION OF ROTATORY MOTIONS.

370. *Proposition I.* If a system be supposed to turn at the same time about three rectangular axes of the co-ordinates OX, OY, OZ (Fig. 36.), with the angular velocities p, q, r, respectively, so that, in the instant dt, it describes, about these axes, the very small angles $p\,dt$, $q\,dt$, $r\,dt$; to express the change of place of any point M of the system in the time dt.

First, consider the angular motion $r\,dt$ about the axis OZ from X towards Y. Let OQ be the projection of OM on the plane XOY. The point Q will describe, in the instant dt, the small arch $Qq = OQ.r\,dt$; and the point M will describe an arc equal and parallel to Qq. The co-ordinates of the point M are $OP = x$, $PQ = y$, $QM = z$; and the similar triangles OPQ, qQr, give

$$OQ : Qq :: OP : qr :: PQ : Qr,$$

$$\text{or} \quad 1 : r\,dt :: x : dy :: yz : -dx.$$

Whence, for this rotation, we shall have

$$dx = -ry.dt; \quad dy = rx.dt; \quad dz = 0.$$

Next, consider the rotation $p.dt$ about the axis OX, from Y towards Z. Let OC be the projection of OM on the plane YOZ. The point C will describe the small arc $Cc = OC.p\,dt$; and the point M will describe an arc equal and parallel to Cc. And the co-ordinates of M being $OB = y$, $BC = z$, $CM = x$, the similar triangles OBC, cCs, give

$$OC : Cc :: OB : cs :: BC : Cs,$$

$$\text{or} \quad 1 : p\,dt :: y : dz :: z : -dy.$$

Whence, for this rotation, we have

$$dx = 0; \quad dy = -pz.dt; \quad dz = py.dt.$$

Lastly, considering the rotation about the axis OY, from Z towards X, let ON be the projection of OM on the plane ZOX. The point N will describe the small arc $Nn = ON . q\,dt$; and the point M will describe an arc equal and parallel to Nn: and the co-ordinates of M being $OL=z$, $LN=x$, $NM=y$, we shall have, from the similar triangles OLN, $n\,Nt$,

$$ON : Nn :: OL : nt :: LN : Nt,$$
$$\text{or}\quad 1 : q\,dt :: z : dx :: x : -dz\,;$$

wherefore, for this rotation, we have

$$dx = qz.dt;\quad dy = 0;\quad dz = -\,qx.dt.$$

And, if all the three rotations take place together, we shall have

$$(C)\quad \begin{aligned} dx &= dt\,(qz - ry),\\ dy &= dt\,(rx - pz),\\ dz &= dt\,(py - qx). \end{aligned}$$

371. *Coroll.* 1. Through O draw the straight line OG, determined by this proportion,

$$x : y : z :: p : q : r.$$

For every point of this straight line $dx=0$, $dy=0$, $dz=0$. Wherefore, for the instant dt, OG will remain unmoved, and the whole system will only revolve around it. From this property OG is called the *Instantaneous Axis of Rotation.*

372. The position of this instantaneous axis of rotation is easily determined.

Let f, g, h, be the angles which it makes with the axis OX, OY, OZ; and put $\sqrt{(p^2 + q^2 + r^2)} = V$. Then

$$\cos. f = \frac{p}{V};\ \cos. g = \frac{q}{V} = \cos. h = \frac{v}{V}.$$

373. *Coroll.* 3. The angular velocity of this instantaneous rotation about OG will be V.

For, drawing PK perpendicular to OG, we shall have $PK = OP\,.\sin.f = (372)\,\frac{x}{V}\sqrt{(q^2+r^2)}$. Now, for the point P, we have $y = 0$, $z=0$; and consequently (370.) $dx = 0$,

$dy = rx\,dt$, $dz = -qx\,dt$. Wherefore, the velocity of the point P will be

$$\frac{\sqrt{(dx^2+dy^2+dz^2)}}{dt} = x\sqrt{(q^2+r^2)}.$$

And the angular velocity about OG will be

$$\frac{x\sqrt{(q^2+r^2)}}{PK} = V.$$

374. *Coroll.* 4. Three rotations $p\,dt$, $q\,dt$, $r\,dt$, which take place together, about three rectangular axes, are equivalent to a single rotation, $dt\sqrt{(p^2+q^2+r^2)}$ about an axis which makes, with the three first, angles having for their cosines

$$\frac{p}{\sqrt{(p^2+q^2+r^2)}},\quad \frac{q}{\sqrt{(p^2+q^2+r^2)}},\quad \frac{r}{\sqrt{(p^2+q^2+r^2)}}.$$

And, vice versâ, a rotation $V.dt$, about a given axis, is equivalent to three simultaneous rotations $V\,dt\cos.f$, $V\,dt\cos.g$, $V\,dt\cos.h$, about three rectangular axes, which make, with the given axis, angles having for cosines f, g, h.

375. *Proposition II.* If the axes OX, OY, OZ (Fig. 36.), be supposed to be fixt in the system, so that, the three rotations taking place as in the preceding proposition, each of the axes revolves, together with the system, about the other two; to express the change of place of the three axes in the instant dt.

These changes of position may conveniently be exhibited by means of Spherical Trigonometry. With the centre O (Fig. 37.) and with a radius $= 1$, suppose a sphere ABT to be described in solid space, and let X, Y, Z, be the points of its surface, through which pass the axes OX, OY, OZ. These axes being at right angles to each other, the arcs XY, YZ, ZX, will be quadrants of great circles, and will cut one another at right angles. Let there be taken, at pleasure, a great circle of the sphere ATB, and in it let there be likewise assumed any point T. Also, the arcs of great circles TX, TY, TZ, being supposed to be drawn, let $TX = l$, $TY = m$, $TZ = n$, and the angles $BTX = \lambda$, $BTY = \mu$, $BTZ = \nu$. Then, the three rotations $p\,dt$, $q\,dt$, $r\,dt$, about the axes OX, OY, OZ, taking place conjointly, it is required to find what space the points X, Y, Z, will describe, in the instant dt, and how the angles l, m, n, λ, μ, ν, will be varied.

First, let the motion of the point X be investigated. By the rotation $p\,dt$, which is made on the pole X itself, it will not move at all: whence, for this rotation we have

$$dl = 0\,;\ d\lambda = 0.$$

For the rotation $q\,dt$, which is made on the pole Y, the point X will describe a small arc $XQ = q\,dt$, perpendicular to YX: and drawing QR perpendicular to TX, we shall have

$$\begin{aligned} XR &= XQ\,.\,\text{cos. } TXQ, \\ &= -\ XQ\,.\,\text{cos. } TXZ = -\ q\,dt\,\frac{\text{cos. } n}{\text{sin. } l}\,, \\ QR &= XQ\,.\,\text{sin. } TXQ \\ &= XQ\,.\,\text{cos. } TXY = q\,dt\,.\,\frac{\text{cos. } m}{\text{sin. } l}\,. \end{aligned}$$

But $XR = -\,dl$, and the small arc QR is the measure of the angle QTX in a circle which has for radius sin. TX; whence $QR = -\ d\lambda$ sin. l. Wherefore, for this second rotation,

$$dl = q\,dt\,\frac{\text{cos. } n}{\text{sin. } l}\,;\ d\lambda = -\,q\,dt\,\frac{\text{cos. } m}{\text{sin.}^2\, l}\,.$$

In like manner, for the rotation $r\,dt$, which is made about the pole Z, we shall find

$$dl = r\,dt\,\frac{\text{cos. } m}{\text{sin. } l.}\,;\ d\lambda = -\,r\,dt\,\frac{\text{cos. } n}{\text{sin.}^2\, l}\,.$$

And when all the three rotations take place together, we shall have, for the movement of the pole X, the two equations

$$\begin{aligned} dl\ \text{sin. } l &= dt\,(q\,.\,\text{cos. } n - r\ \text{cos. } m), \\ d\lambda\ \text{sin.}^2\, l &= -\ dt\,(q\,.\,\text{cos.} m + r\ \text{cos. } n). \end{aligned}$$

In the same manner may be expressed the changes in place of the other two poles Y, Z. Whence we shall have for the angles l, m, n, λ, μ, ν, these six equations,

$$(D)\ \begin{cases} dl\,.\,\text{sin. } l = dt\,(q\ \text{cos. } n - r\ \text{cos. } m), \\ dm.\text{sin.} m = dt\ (r\ \text{cos. } l - p\ \text{cos. } n), \\ dn.\text{sin. } n = dt\,(p.\text{cos.} m - q\,.\,\text{cos. } l). \end{cases}$$

$$(E)\begin{cases} d\lambda\,.\,\sin.^2 l = -dt\,(q\,.\cos.\,m + r\cos.\,n), \\ d\mu\,.\,\sin.^2 m = -dt\,(r\,.\cos.\,n + p\cos.\,l), \\ d\nu\,.\,\sin.^2 n = -dt\,(p\,.\cos.\,l + q\cos.\,m). \end{cases}$$

376. **Scholium 1.** Of the three angles l, m, n, it will be sufficient to obtain two from the equations (D); for the third will be known from

$$\cos.^2 l + \cos.^2 m + \cos.^2 n = 1:$$

and of the other three λ, μ, ν, it will be sufficient to obtain one from the equations (E); for the other two will be known from

$$\cos.\,(\mu - \lambda) = -\cot.\,l\cot.\,m,$$
$$\cos.\,(\nu - \mu) = -\cot.\,m\cot.\,n.$$

377. **Scholium 2.** By the formulas here collected, if we know the three velocities p, q, r, with which the system revolves about three rectangular axes, we shall know the situation of any point of it at any instant; either by means of the equations (C) if these three axes be fixt in space, or else by the equations (D), (E), if they be moveable along with the system.

CHAP. XXIII.

ON THE MOVEMENT OF A FREE BODY ACTED UPON BY SEVERAL FORCES.

378. ***Proposition I.*** The forces which act upon each point of the system being given, to determine the initial movement.

Let each element dM be referred to three rectangular axes, which pass through the centre of gravity of the body or system, and let the forces acting upon that element be reduced (30.) to three P, Q, R, parallel to the three axes. What will be the initial motion of the body is easily collected from the Propositions laid down in the two preceding Chapters.

First, it follows from Art. 364, 365, that the centre of gravity will move, in the direction of x, with the velocity $\frac{\Sigma . P}{M}$; in the direction of y, with the velocity $\frac{\Sigma . Q}{M}$; and in the direction of z with the velocity $\frac{\Sigma . R}{M}$: so that the absolute velocity will be (19.)

$$u = \frac{1}{M} \surd \{(\Sigma . P)^2 + (\Sigma . Q)^2 + (\Sigma . R^2)\},$$

and its direction will make (27.) with the axes of x, y, z, respectively, angles the cosines of which are $\frac{\Sigma . P}{Mu}$, $\frac{\Sigma . Q}{Mu}$, $\frac{\Sigma . R}{Mu}$.

Secondly, the body will revolve (364.) about its centre of gravity; and we are next to find about what axis it will revolve, and with what angular velocity.

Let F, G, H, be the sums of the momentums of the forces P, Q, R, to turn the body about the axes of x, y, z; and let

A, B, C, be the momentums of inertia of the body itself referred to the same three axes respectively. The angular velocities dp, dq, dr, with which the body will begin to turn about these three axes will be (365.) $\frac{Fdt}{A}$, $\frac{Gdt}{B}$, $\frac{Hdt}{C}$. If, then, we put

$$\sqrt{\left(\frac{F^2}{A^2}+\frac{G^2}{B^2}+\frac{H^2}{C^2}\right)} = \Omega,$$

the instantaneous axis of initial rotation will make (374.) with the three axes, angles the cosines of which are $\frac{F}{A.\Omega}$, $\frac{G}{B.\Omega}$, $\frac{H}{C.\Omega}$; and the incipient angular velocity will be $d\omega = \Omega.dt$.

379. Scholium 1. Hitherto we have determined the *initial* movement of the body. But as their mutual connexion does not leave each of the elements of the body at liberty to preserve the velocity and the direction first acquired, it is necessary, in order to determine the progress of the motion, to recur to that proposition (299.) which teaches us how to assign the movements, as they are affected by mutual actions amongst the parts composing the system.

It is to be recollected, therefore, that x, y, z, being the coordinates of the element dM, the velocities with which this element moves in the directions of the ordinates, will be (206.) $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$. The increments of the velocities impressed in the instant dt, by the accelerating forces P, Q, R, (206.) are Pdt, Qdt, Rdt. The increments of the velocities, which obtain effectively, are expressed by $d.\frac{dx}{dt}$, $d.\frac{dy}{dt}$, $d.\frac{dz}{dt}$. Now (300.) the system of forces due to the impressed velocities, must be in equilibrium with the system of forces due to the effective velocities, supposing them to act in a contrary direction. Wherefore, if each element dM were acted upon by the three forces

$$dM\left(Pdt-d.\frac{dx}{dt}\right),\ dM\left(Qdt-d.\frac{dy}{dt}\right),\ dM\left(Rdt-d.\frac{dz}{dt}\right)$$

acting respectively in the directions of the three axes, the whole system must be in a state of equilibrium.

380. Scholium 2. For this equilibrium two conditions (123.) are required. The first is that the sum of the forces parallel to each of the three axes must $= 0$. Hence, making the increment dt constant, we shall have the three equations

$$\int P\,dM = \int dM\,.\frac{ddx}{dt^2},$$

$$(A)\quad \int Q\,dM = \int dM\,.\frac{ddy}{dt^2},$$

$$\int R\,dM = \int dM\,.\frac{ddz}{dt^2}\,.$$

The second condition is, that the sum of the momentums of the forces, to turn the body about each of the three axes, must $= 0$. Wherefore (114.) for the rotation about the axis of x, we shall have

$$\int z\,dM\left(Q\,dt - \frac{ddy}{dt}\right) - \int y\,dM\left(R\,dt - \frac{ddz}{dt}\right) = 0;$$

$$\text{or, } \int dM\,dt\left(Qz - Ry\right) = F.dt = \int y.dM.\frac{ddz}{dt} - \int z\,dM.\frac{ddy}{dt}.$$

And, operating in a similar manner for the other two axes, there will result the three equations

$$F\,dt = \int y\,dM\,.\frac{ddz}{dt} - \int z\,dM\,.\frac{ddy}{dt},$$

$$(B)\quad G\,dt = \int z\,dM\,.\frac{ddx}{dt} - \int x\,dM.\frac{ddz}{dt},$$

$$H\,dt = \int x\,dM\,.\frac{ddy}{dt} - \int y\,dM.\frac{ddx}{dt}\,.$$

As, therefore, the equilibrium of a solid system is determined by means of six equations (124.), so by six equations, and no more, is its motion determined. By the three first (A), the progressive motion of the centre of gravity is determined; by the three others (B), is determined the rotation of the body about the centre of gravity, as in the two following propositions we shall separately shew.

381. *Proposition II.* To determine, at any given time, the position of the centre of gravity of a body, the forces, which act upon each point of it, being given. After the time t, let the co-ordinates of the centre of gravity be X, Y, Z; by the property of that centre we have

$$MX = \int x.dM;\ MY = \int y.dM;\ MZ = \int z.dM.$$

Hence, differentiating twice, and dividing each time by dt,

$$\frac{M.ddX}{dt^2} = \int dM.\frac{ddx}{dt^2},$$
$$\frac{M.ddY}{dt^2} = \int dM.\frac{ddy}{dt^2},$$
$$\frac{M.ddZ}{dt^2} = \int dM.\frac{ddz}{dt^2}.$$

Substituting these values in the equations (A), we have the equations

$$\frac{ddX}{dt^2} = \frac{\int P.dM}{M};\ \frac{ddY}{dt^2} = \frac{\int Q.dM}{M};\ \frac{ddZ}{dt^2} = \frac{\int R.dM}{M},$$

by which the motion of the centre of gravity is completely determined.

382. *Coroll.* These equations are precisely the same as those by which is determined the motion of a point, acted upon (231.) by the accelerating forces $\frac{\int P.dM}{M}$, $\frac{\int Q.dM}{M}$, $\frac{\int R.dM}{M}$. Now if the whole mass M of the body were collected in the centre of gravity, and if the forces P, Q, R, which act upon the elements dM of the mass, were all transferred to the centre of gravity, it is manifest that this centre would then be acted upon by the accelerating forces $\frac{\int P.dM}{M}$, $\frac{\int Q.dM}{M}$, $\frac{\int R.dM}{M}$. It may be concluded, therefore, that the centre of gravity of a free body moves in the same manner as it would do, if the whole mass were re-united there, and if all the forces were immediately applied to that point.

383. SCHOLIUM. In the first Chapters of this Book, whilst we were treating of the most simple motions of bodies of any kind, and particularly of heavy bodies, we considered the bodies themselves as points or atoms, abstracting altogether (225.)

from their mass and their figure. We now see that, in fact, the *progressive* motion of a free body, is always the same as if the body were reduced to a single point, and that neither the mass nor the figure produce any change in it. The *progressive* motion may, however, sometimes be accompanied by a *rotatory* motion, which is determined by the following proposition.

384. *Proposition III.* To determine, at any given time, the position of any point of a system, with respect to the centre of gravity, the forces which act upon each point of the system being given.

At the end of the time t, let the centre of gravity be found at O (Fig. 36.).

Let the point O be taken for the origin of the co-ordinates, x, y, z, and the three principal axes, drawn through it, for the axes of these co-ordinates. Also, let p, q, r, be the angular velocities with which the body turns about each of the three axes. To determine p, q, r, we may proceed thus.

In the equations (B) of Art. 380, instead of the differentials of the co-ordinates, dx, dy, dz, put their values deduced from the equations (C) of Art. 370. And, since OX, OY, OZ, are principal axes, we have (330.) $\int xy\,.\,dM = 0$, $\int xz\,.\,dM = 0$, $\int yz.dM = 0$; whence all the terms will disappear, which, under the integrating sign, contain either $xy\,dM$, $xz\,dM$, or $yz\,dM$, multiplied by any function of p, q, r; because p, q, r, are functions of no other variable quantity than t, and therefore go out of the sign of integration. Further, calling, as before, A, B, C, the momentums of inertia of the body with respect to the three axes OX, OY, OZ, we have

$$\int dM(y^2+z^2)=A;\ \int dM(x^2+z^2)=B;\ \int dM(x^2+y^2)=C.$$

The substitution above directed, being therefore completed accordingly, will give, in the last place, the three equations, of a very simple form,

$$(B')\quad \begin{aligned} F\,.\,dt &= A\,.\,dp + (C - B)\,qr\,.\,dt, \\ G\,.\,dt &= B\,.\,dq + (A - C)\,pr\,.\,dt, \\ H\,.\,dt &= C\,.\,dr + (B - A)\,pq\,.\,dt; \end{aligned}$$

by which may be determined the three velocities of rotation p, q, r; and thence the situation of every point of the system.

For, when the values of p, q, r, are expressed in terms of t, by substituting them in the equations (D), (E), of Art. 375, and then integrating those equations, we shall know what, at any time, will be the situation of the three principal axes OX, OY, OZ. Afterwards, substituting the same values in the equations (C) of Art. 370, and then integrating them, we shall have, expressed in terms of t, the co-ordinates x, y, z, of any point referred to these three axes, of which the situation is already determined.

385. *Coroll.* The three velocities p, q, r, being known, we shall also know (372, 373.) for any instant, about what axis, and with what velocity, the body revolves.

386. *Proposition IV.* To determine the rotation of a body not acted upon by any accelerating force.

In this case, $F = G = H = 0$; and if we put

$$\frac{B-C}{A} = L,\ \frac{C-A}{B} = M,\ \frac{A-B}{C} = N,$$

the three equations (B), of Art. 384, become

$$dp = Lqr.dt;\ dq = Mpr.dt;\ dr = Npq.dt.$$

Put $pqr.dt = d\phi$, and we have

$$p.dp = L.d\phi;\ q.dq = M.d\phi;\ r.dr = N.d\phi;$$

and, integrating,

$$p^2 = 2L\phi + a^2;\ q^2 = 2M\phi + b^2;\ r^2 = 2N\phi + c^2,$$

a, b, c, being the initial values of p, q, r: hence

$$dt = \frac{d\phi}{\sqrt{(2L\phi + a^2).(2M\phi + b^2).(2N\phi + c^2)}}.$$

Integrating this equation, so that $t = 0$ may give $\phi = 0$, we shall know, at any instant, the value of ϕ, and thence also those of the three velocities p, q, r.

It remains to determine the position of the principal axes, deducing from the equations (D) (375.), the angles l, m, n, and from the equations (E) the angles λ, μ, ν. If, for the sake of abridgement, we write

$$\sqrt{(A^2a^2 + B^2b^2 + C^2c^2)} = K,$$

the equations (D) will be satisfied, by putting

$$\cos. l = \frac{Ap}{K};\ \cos. m = \frac{Bq}{K};\ \cos. n = \frac{Cr}{K}:$$

And these values, substituted in the equations (D) render them identical with the equations (B') (384.).

Lastly, it will suffice (376.) to resolve the first of the equations (E), in order to deduce from it the angle λ. Now this first equation, after the substitution of the preceding values of l, m, n, becomes

$$d\lambda\,.(A^2p^2 - K^2) = K\,.dt\,.(Bq^2 + Cr^2):$$

where, if in the place of p, q, r, we put their values already found in terms of t, we shall have a separated differential equation between λ and t, the integration of which will give, for each instant, the angle λ, and thence also the other two.

387. SCHOLIUM. The values of the angles l, m, n, may seem incomplete, inasmuch as the arbitrary constant quantities a, b, c, are wanting; which must afterwards be determined, so that $t = 0$ shall give the initial values of those angles. But this is not necessary here; because the initial values are themselves arbitrary, it being allowable to fix at pleasure the point T in the sphere (375.) whence spring the arcs l, m, n. The proposed solution will, therefore, be complete, by so fixing the above-mentioned point, that, in the beginning of motion, we may have

$$\cos.\, l = \frac{Aa}{K};\ \cos.\, m = \frac{Bb}{K};\ \cos.\, n = \frac{Cc}{K}.$$

CHAP. XXIV.

ON THE ROTATION OF A FREE BODY ABOUT THE PRINCIPAL AXES.

388. ***Proposition I.*** If a body begin to revolve about a principal axis, it will continue to revolve perpetually and equally about that axis.

Let the body begin to turn about the axis OX, with the angular velocity a; we shall have (386.) $b = c = 0$, and

$$dt = \frac{1}{2\sqrt{MN}} \cdot \frac{d\phi}{\phi\sqrt{(2L\phi + a^2)}};$$

or, substituting for $2L\phi + a^2 = p^2$,

$$dt = \frac{1}{\sqrt{MN}} \cdot \frac{dp}{p^2 - a^2};$$

whence, integrating, we deduce

$$C + t\sqrt{MN} = \frac{1}{2a} \log. \frac{p - a}{p + a}, \text{ or}$$

$$\frac{p - a}{p + a} = e^C \cdot e^{2at\sqrt{MN}}.$$

But it is necessary to determine the constant quantity C, so that when $t = 0$, p may $= a$: wherefore, $e = 0$. Hence at any time t, we have $p = a$; therefore $\phi = 0$, and $q = r = 0$.

389. Scholium 1. This singular property of the principal axes, that the impressed rotation about them continues perpetual and uniform, we had before ascertained in another manner; it having been demonstrated (347.) that the centrifugal forces of the

elements of a body revolving about such axes, counterbalance each other; so that the axis is equally drawn by them on every side, and does not incline toward one side more than toward the other. Thus an axis of this kind remains unmoved, without the aid of any support.

390. Scholium 2. We shall now investigate what takes place, if the axis about which the rotation commences, be not the principal axis OX, but remote from it by a very small distance. Hence we shall understand further what will take place, when, the body revolving about a principal axis, its rotation is disturbed by a very small shock.

And since, generally speaking, of the three principal axes there are two, to which belongs the greatest, and the least, momentum of inertia, whilst of the third the momentum of inertia is neither a maximum nor a minimum (322.) we shall separate these cases in the two following propositions.

391. *Proposition II.* If a body begin to revolve about an axis very near to that principal axis, to which belongs the greatest or the least momentum of inertia, the axis of rotation will go on continually oscillating about the principal axis, always remaining very near to it.

In this case, the initial velocities b, c (386.), will no longer be equal to nothing, but will be very small in respect of a. Hence the equation between t and ϕ will give ϕ a very small quantity, and therefore p will be very nearly $= a$. Putting $p = a$ in the two last of the equations, (B') (384), and then differentiating them, their combination will give

$$\frac{ddq}{dt^2} - a^2 MNq = 0; \quad \frac{ddr}{dt^2} - a^2 MNr = 0;$$

two linear equations, which, integrated so that $t = 0$ may give $q = b$, and $r = c$, will make known the values of q and r, expressed in terms of t.

Now, by the hypothesis, the momentum of inertia A, being a maximum, or else a minimum, will either be greater than each of the other two B, C, or less than each of them. In both cases, it is evident (386.) that M and N will have contrary signs. Let, therefore, $N = -Mk^2$, and the two equations will become

$$\frac{ddq}{dt^2} + a^2 M^2 k^2 q = 0; \quad \frac{ddr}{dt^2} + a^2 M^2 k^2 r = 0,$$

which, integrated as above directed, will give

$$kq = c \sin. aMkt + bk \cos. aMkt;$$
$$r = c \cos. aMkt - bk \sin. aMkt.$$

These values of q and r remain always very small, whatever may be this increase of t, and go on continually vibrating betwixt the limits $q = b$, $kq = c$; $r = c$, $r = bk$. Whence it may be easily inferred (374.) that the axis of rotation goes on perpetually oscillating about the principal axis, remaining always very near to it; and that the angular velocity of rotation is very nearly constant, being liable only to very small periodical alterations.

392. *Proposition III.* If a body begin to revolve, about an axis very near to that principal axis to which belongs neither the greatest, nor the least momentum of inertia, the axis of rotation removes itself, indefinitely, from the principal axis, and the velocity of rotation does not remain constant.

The momentum of inertia A being, by the hypothesis, intermediate between the other two B, C, it is plain that M and N (386.) will have the same sign. Let $N = Mk^2$. The equations

$$\frac{ddy}{dt^2} - a^2 M^2 k^2 q = 0, \quad \frac{ddr}{dt^2} - a^2 M^2 k^2 r = 0,$$

when integrated, will give

$$2kq = (bk + c)e^{aMkt} + (bk - c)e^{-aMkt},$$
$$2r = (bk + c)e^{aMkt} - (bk - c)e^{-aMkt}.$$

These values, as t increases, vary beyond all limits. Whence it may readily be inferred, that the axis of rotation may remove itself indefinitely from the principal axis, and that the velocity of rotation may increase, or be diminished, indefinitely.

393. *Coroll.* From what has been said it may be collected, that although it is always true that the revolution of a body about

any of its principal axes is, by its own nature, perpetual and uniform; if nevertheless, it happen that, from any new impulse, the axis of revolution have any small nutation, when this axis is one of the two to which belongs the greatest or the least momentum of inertia, then the uniformity of the rotatory motion will not be disturbed, further than that the axis of rotation will go on vibrating about its first position. This will not be so, if the revolution be made about the third axis. Wherefore, the rotation is stable about the two first axes, unstable about the third.

ON MOTION.

SECTION III.

ON THE MOTION OCCASIONED BY IMPACT.

CHAP. XXV.

ON DIRECT AND CENTRAL IMPACT.

394. IF two bodies, going on with a progressive motion, come in contact with one another, and if their centres of gravity move in the same straight line, which is perpendicular to a plane touching both bodies in the point of contact, the impact is said to be *direct* and *central.*

But if the straight line described by the centre of gravity of one of the bodies, is perpendicular to the tangent plane, but yet does not pass through the centre of gravity of the other body, the impact is said to be *direct* but *eccentric.*

If, lastly, the path of the centre of gravity of one of the bodies is not perpendicular to the tangent plane, the impact is said to be *oblique.*

We shall, first, consider direct and central impact.

395. ***PROPOSITION.*** If the mass A, moving with the velocity V, strike the mass B, moving, in the same direction, with the less velocity v, the two masses, after direct and central impact, will move on together; and their common velocity will be

$$V - \frac{B(V-v)}{A+B}, \text{ or } v + \frac{A(V-v)}{A+B}, \text{ or } \frac{AV+Bv}{A+B}.$$

It is plain that the two masses will, in consequence of the impact, acquire an equal velocity: for if the mass A preserved a velocity greater than that acquired by the mass B, this latter would impede the motion of A; and the impact would continue, until the mass B, having acquired a velocity as great as that retained by A, would no longer serve as an impediment to its progress.

Let then x be this common velocity; the forces of the two masses, which (133.) before the impact were $A.V$, $B.v$, will, after the impact, become $A.x$, $B.x$. Wherefore, (300.) the forces $A.V$, $B.v$, must be in equilibrium with the forces $-Ax$, $-Bx$: whence,

$$A(V-x)+B(v-x)=0,$$

from which equation are deduced the assigned values of x.

If the body struck be at rest, $v=0$; and, if the two bodies meet each other, v becomes negative.

396. *Coroll.* 1. The change of force is equal in both the bodies; that of the velocities is inversely proportional to the mass.

397. *Coroll.* 2. The velocity of the common centre of gravity of the two bodies, and the sum of their forces, remains the same after impact as it was before.

CHAP. XXVI.

ON THE IMPACT OF ELASTIC BODIES.

398. **Hypothesis.** There are bodies which, by impact, suffer compression, and, after impact, endeavour to resume their original form, with a force proportional to the impact itself: so that this force of restitution gives them, in a contrary direction, a determinate part of the force, and consequently of the velocity, which they had lost in the impact.

399. This force of restitution is called *Elasticity*, which is said to be *perfect*, if the body endeavours to resume its form with precisely as much force as its loses in the impact; and *imperfect*, if the body endeavours to resume its form with a less force.

If bodies suffer no compression at all from impact, they are called *hard;* and if, when compressed, they exert no force whatever to recover their original form, they are called *soft*.

400. ***Proposition.*** If the elastic mass A, proceeding with the velocity V, strike with a direct and central impact, the elastic mass B, moving with the velocity v; and if the fraction p express that part of the velocity lost in impact, which is re-acquired by the body, from its elasticity, in a contrary direction, then, after the impact, the velocities will be

$$\text{of the mass } A \ldots\ldots V - (1 + p) \, . \, \frac{B(V - v)}{A + B},$$

$$\text{of the mass } B \ldots\ldots v + (1 + p) \, . \, \frac{A(V - v)}{A + B}.$$

Calling x the common velocity which the bodies A and B would acquire by the sole effect of impact, A will have lost by impact the velocity $V - x$, and B the velocity $v - x$. Wherefore, after impact and the successive restitution, the velocity of A (398.) will be $x - p\,(V - x)$, and the velocity of B will be $x - p\,(v - x)$: where, putting for x its value (395.) $\dfrac{AV + Bv}{A + B}$; there will result the velocities above expressed.

401. *Coroll.* 1. If we make $p = 0$, we return to the case of non-elastic bodies, which is the subject of the preceding Chapter: if we make $p = 1$, we have the case of *perfectly* elastic bodies. In general, whatever be the value of p, the properties, enuntiated in Art. 396, 397, will be found to obtain.

402. *Coroll.* 2. The product of the mass of a body by the square of its velocity, is called the body's *Vis Viva.* And if the loss of vis viva, which takes place in impact, be calculated, it will be found to be

$$(1 - p^2)\,\frac{A\,.\,B\,.(V - v)^2}{A + B}\,;$$

whence it is nothing in the impact of perfectly elastic bodies, and it is greatest in the impact of non-elastic bodies.

403. *Coroll.* 3. The difference of the velocities, or the relative velocity, which before the impact was $V - v$, becomes, after the impact, $-p\,(V - v)$; so that, if the two bodies are perfectly elastic, the relative velocity is the same both before and after the impact.

404. *Coroll.* 4. If the two masses are equal and perfectly elastic, they exchange, by the impact, their velocities.

405. *Coroll.* 5. An elastic body striking, directly, with the velocity V, an immoveable obstacle, rebounds, in the same line, with a velocity $= pV$. For, then, it is the same as if the mass B, of the body struck, were infinite, and its velocity $= 0$; in which case the velocity of the striking body A becomes $-pV$.

406. *Coroll.* 6. Let there be a series of elastic masses decreasing in a geometric progression, of which the exponent is q: let the first mass, moving with the velocity V, strike the second at rest; let this, with the velocity acquired from the impact, strike the third, and so on: it is required to find the velocity which will be communicated to the n^{th} mass.

Applying the formula of Art. 400, we shall find that the second mass receives the velocity $q\,.\,\frac{1 + p}{1 + q}\,.\,V$; that the third receives the velocity $\left(q\,.\,\frac{1 + p}{1 + q}\right)^2 .\,V$; and that the velocity of the n^{th} mass is $\left(q\,.\,\frac{1 + p}{1 + q}\right)^{n - 1} V$.

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CHAP. XXVII.

ON ECCENTRIC IMPACT.

407. *Proposition.* Let the non-elastic mass A, moving with the velocity V, strike with an eccentric impact, the non-elastic mass B, moving with the velocity v, and in a direction parallel to that of A: it is required to find the motion of both the masses after impact.

Let MOE, MRD (Fig. 35.) be the sections of the two bodies A, B, made by a plane passing through their centres of gravity T, G, and through the direction TA of A's motion; let the distance of this direction, from the centre of gravity G of the mass B, be $GA = a$: let x be the velocity which remains to the mass A after impact; and since the other mass B will (364.) after impact, have two motions, the one progressive, the other rotatory about the axis GX, let u be the velocity of the former, and ω the angular velocity of the latter.

First, then, the forces $A.V$, $B.v$, must (300.) be in equilibrium with the forces $-A.x$, $-B.u$; whence

$$A(V - x) + B(v - u) = 0.$$

The body B is urged to revolve, by the force which the body A loses in the impact, which is $A(V - x)$. Wherefore, calling S the momentum of inertia of the body B, in respect to the axis GX, we shall have (365.),

$$\omega = \frac{aA(V - x)}{S}.$$

Lastly, from what was said in Art. 395, it appears that as much velocity ought to remain to the mass MOE, as those points of the mass MRD acquire, which are in the line TA, along which the body MOE proceeds in its motion; so that the remaining velocity of the striking body must be that which the point A, of the mass struck, acquires. But the remaining velocity of the striking body is $= x$; and the velocity which the point A acquires is (367.) $= u + a\omega$. Wherefore,

$$x = u + a\omega.$$

We have, therefore, three equations, from which may be found the three unknown quantities x, u, ω.

408. *Coroll.* 1. The problem will scarcely be more difficult, if one, or both the bodies be elastic. The force of elasticity restoring to the bodies in a contrary direction, a velocity proportional to that which they lose from the impact, it will suffice, first, to find the three velocities x, u, ω, as they would be, if there were no elasticity; and then to substitute for them, respectively,

$$x - p(V - x);\ u - p(v - u);\ \omega + p\omega.$$

409. *Coroll.* 2. If the body struck, is connected by an inflexible line, with an immoveable axis, it cannot have, or acquire from the impact, any progressive motion. There remain, then, only two unknown quantities x, ω; to determine which, referring the momentums to the axis of rotation, we have, as above, the two equations

$$\omega = \frac{A a (V - x)}{S};\ x = a\omega.$$

CHAP. XXVIII.

ON OBLIQUE IMPACT.

410. ***Proposition.*** To find the motion of two given masses, moving with given velocities, and in given directions, after oblique impact.

Resolve the velocity of each of the masses into two, the one perpendicular to the plane touching both the masses in their point of contact, the other parallel to that plane. The first occasions the impact and undergoes those changes, which we have before shewn how to determine. The second remains unaltered; because, by it, the bodies do not strike, nor mutually act upon, each other. Compounding, therefore, this latter with the former, modified according to the laws of direct impact, we shall have the velocities, and the directions, of the two bodies after the impact.

411. *Coroll.* 1. By way of example, let this problem be proposed. The velocity, and the angle of incidence, with which an elastic sphere strikes an immoveable plane being given, to find the velocity and the angle of reflection.

Let u be the velocity, and a the angle of incidence. Resolve the velocity u into two; the one perpendicular to the plane, which will be $= u \sin. a$; this causes the impact, in consequence of which, the sphere would rebound perpendicularly (405.) with a velocity $= pu . \sin. a$: the other parallel to the plane, which will be $= u . \cos. a$; and this remains unaltered.

Compounding, then, these two velocities $pu . \sin. a$, $u \cos. a$, the directions of which are perpendicular to each other, the velocity of reflection will be found to be

$$u \sqrt{}(\cos.^2 a + p^2 . \sin.^2 a).$$

And, if β be the angle of reflection,

$$\tan. \beta = \frac{pu \sin. \alpha}{u \cos. \alpha}; \text{ or } \tan. \beta = p \,.\, \tan. \alpha.$$

412. *Coroll.* 2. Hence, if the elasticity be perfect, the velocity, and the angle of reflection, will be equal to the velocity, and to the angle of incidence.

◆

APPENDIX.

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APPENDIX.

PROBLEMS IN MECHANICS.

1. Of three forces, which, acting upon a given point, keep it at rest, the quantities of the first and third, and the directions of the first and second, being given, to find the quantity of the second force, and the direction of the third.

2. Three forces being each of them given as to quantity, and their directions being each to pass through a given point, to find a point, which, if the forces be made to act upon it, shall be at rest: the three given points not being in the same straight line, and any two of the forces being greater than the third.

3. A body, acted upon by three forces, cannot be at rest, unless the directions of the forces be in the same plane and meet in one point, and unless each of them passes through the angle made by the other two.

4. A point, placed in the centre of a sphere, and acted upon by more than two forces, cannot be at rest, unless the directions extend, if they be in the same plane, over more than a semicircle; or, if they be in different planes, over more than a hemisphere.

5. There being given the directions of four forces lying in different planes, and acting upon a given point, to determine the quantities of the forces, when they keep the given point at rest.

6. If the directions of more forces than four be given, the forces lying in different planes and acting upon a given point, and if it be required to determine their quantities, so that they may keep the given point at rest, the problem is either indeterminate or impossible.

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7. If any number of forces S, S', S''.... act upon a free point in the directions s, s', s''...., and if r be the direction of their resultant R, then,

$$R\,dr = S\,ds + S'ds' + S''ds''\ldots\ldots$$

8. If a system of points A, B, C.... be acted upon by the forces S, S', S''.... respectively in the directions s, s', s'', when there is an equilibrium,

$$S\,ds + S'ds' + S''ds''\ldots. = 0.$$

9. There being given the quantities and the directions of two forces, acting on the given points of a given lever of any kind, to find the place of the fulcrum, and also the quantity and the direction of the pressure on the fulcrum, when there is an equilibrium.

10. Three weights, A, B, C, being given, and also the weights of the scales D, E, of a balance, to find the place of the fulcrum, so that one of the given weights, as B, may balance the weight A in the scale D, and the weight C in the scale E; the length of the beam, which is supposed to be without weight, being also given: to determine, also, the limitations under which the problem is possible.

11. If the fulcrum be similarly posited in two straight levers, AB, ab, if P, p, be two weights suspended from the ends A, a, of the corresponding arms AC, ac, and if at the ends of the other CB, cb, there be hung two counterpoising weights, W, w, which are to one another as $\overline{AC}^n$ to $\overline{ac}^n$, then, $P : p :: \overline{CB}^n : \overline{cb}^n$.

12. If two unequal straight levers have an arm (B) of the one equal to an arm of the other, and if from the two remaining arms (A), (a), be hung the weights P, p, such that $P : p :: A^n : a^n$, and if (W), (w), be the weights which, hung at the other arms, counterbalance them, then, $W : w :: A^{n+1} : a^{n+1}$.

13. The divisions of a steelyard, which has a given weight suspended from one end, and the *fulcrum* of which is moveable are harmonic proportionals.

14. The powers (P), (P'), being supposed to act at the extremities A, B, of a straight lever AB, having its fulcrum at C, in

directions which meet in the point D, there will be an equilibrium, if $P : P' :: AD . CB : BD . CA$.

15. If two particles of matter, acted upon by forces tending to a given fixt point, and varying as the distances of the particles from that point, balance each other, at the ends of a given lever, in any one position of the lever, they will balance each other in any other position of it, the lever being turned about its fulcrum.

16. The angle at which two planes are inclined to each other being given, and the weights, which balance each other, connected by a string passing over a fixt pully, having its centre in the intersection of the planes, to find the angles which the planes make with the horizon: the two parts of the string being parallel to the two planes, each to each.

17. To determine the angle at which the arm of an anchor ought to be inclined to the shank, in order that it may sink the deepest into the bottom of the sea; the anchor being supposed to be dragged in the direction of the shank.

18. If a given power, applied in the direction of any given tangent to a wheel, support a given weight suspended from the axle, to find the direction and the quantity of the pressure, at the centre of that section of the axle, which passes through the directions of the power and the weight.

19. If a straight lever loaded, at a given point, by a given weight, be carried on the shoulders of two men of given heights, to find the quantity of pressure sustained by each of the men.

20. If a weight is to be carried on a straight lever, having its extremities supported on the shoulders of two men, of given heights, whose strengths are to another in a given ratio, to place the weight so that its pressure on each of the men may be proportional to his strength.

21. If a given heavy cylindrical rod be laid upon an horizontal table with a part, which is less than its half, projecting beyond the table, to graduate the projecting part, so that weights of 1, 2.... $n$lb., hung at the divisions, may be just supported.

22. If three given forces P, Q, R, act at the three angles A, B, C, respectively, of a given triangle ABC, perpendicularly to its plane, to find their resultant,

1st, When $P = Q = R$,

2dly, When $P : Q : R :: BC : AC : AB$,

3dly, When $P : Q : R :: \frac{1}{BC} : \frac{1}{AC} : \frac{1}{AB}$,

4thly, When $P : Q : R :: AB : BC : CA$,

5thly, When $P : Q : R :: \frac{1}{AB} : \frac{1}{BC} : \frac{1}{CA}$.

23. To find the same as in the preceding problem, when the forces act in parallel directions, but not perpendicularly to the plane of the triangle.

24. If two given and equal parallel forces act, toward the same parts, at opposite angles of a given parallelogram, to find the two parallel forces which, acting at the two remaining angles of the figure, shall produce an equilibrium.

25. If two parallel forces, acting at opposite angles of a given parallelogram, be not equal, no two forces, applied at the reremaining angles of the parallelogram, can produce an equilibrium: but the equilibrium may be maintained, by two parallel forces acting at two other points of the perimeter of the figure.

26. If a given parallelogram is to be kept at rest by four parallel forces, acting upon points in its perimeter, of which two act, toward the same parts, at opposite angles of the figure, and if the point, at which the third force acts, be also given, to find the third and fourth forces, and the point of application of the fourth.

27. A given triangle being supposed to be without weight, and to rest on three equal vertical props, placed at its three angles, to find a point in it, to which if a given weight be appended, the pressures on the three props shall be to each other in given ratios.

28. If a heavy body be suspended by a string, the two extremities of which are fastened to two points of the body, it will not rest, unless a vertical, drawn through the centre of gravity of the body, pass through the point of suspension.

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29. If each of two given weights be connected with a given heavy sphere, at given points of its surface, by means of a string passing over a fixt pully, and if, when there is an equilibrium, the directions of the two strings meet, in the sphere's centre, at a right angle, to compare the parts of the sphere's weight sustained by the two strings.

30. If G be the centre of gravity of a given triangle ABC, then $GA : GB : GC :: \sin. BGC : \sin. AGC : \sin. AGB$.

31. To find the locus of the centres of gravity of all the triangles which have a common vertex, a common vertical angle, and equal areas.

32. There may be an indefinite number of triangular pyramids, in which the distances from the centre of gravity to the four angles of the solid are the same.

33. If two given spheres touch one another internally, to find the centre of gravity of the solid included between the convex surface of the less, and the concave surface of the greater sphere.

34. To find the centre of gravity, first, of a given truncated pyramid; secondly, of a given truncated cone; thirdly, of a given segment of a paraboloid, cut off from it by a plane parallel to the axis of the solid.

35. If two given cones be on the same base, to find the centre of gravity of the solid included between their surfaces; first, when the two cones are on contrary sides of the common base; secondly, when they are on the same side of it.

36. A system of heavy bodies being given, and also a point without it, to find the locus of all the points, from each of which if straight lines be drawn to the centres of gravity of the bodies, the sum of the products of each body multiplied by the distance of its centre of gravity from that point, shall be equal to the sum of the products of each of them, multiplied by the distance of its centre of gravity from the given points.

37. If one of the equal sides of an isoceles right-angled triangle, rests on an horizontal plane, and the other side is vertical, to

find the greatest isosceles triangle, which can be described on the hypotenuse, as a base, so that the whole figure, being endued with an uniform gravity, shall not fall.

38. A heavy body cannot be sustained on a given inclined plane, by a force acting at its centre of gravity, unless the direction of the force lie within the supplement of the angle contained by two straight lines, drawn from the centre of gravity, the one perpendicular to the given plane, the other perpendicular to the horizon.

39. If a given beam be suspended from a given point, in a vertical wall, by a string attached to one of its extremities, to place the other extremity of the beam against the wall, so that it shall be at rest.

40. To find the common centre of gravity of three given spheres, consisting of particles attracted to the earth's centre, by forces varying as the distances of the particles from that centre; the centres of the spheres and of the earth being supposed to be in the same given plane.

41. To find the centre of gravity of a given finite straight line, in which the weight of each particle is supposed to vary as its distance from one extremity of the line, so as to be in a given proportion to common gravity at the other extremity.

42. To find the centre of gravity of a triangle, in which the gravity is supposed to be uniform in any parallel to the base, whilst the gravity in any two different parallels, is proportional to their perpendicular distances from a straight line, drawn through the vertex, parallel to the base.

43. To find the centre of gravity of a solid, generated by the revolution of a curve, the equation of which is $y = ax^m$, about its axis, the gravity being uniform in each section parallel to the base, and varying, in different sections, as x^n.

44. To find the centre of gravity of the surface of a hemisphere, the gravity of each particle being supposed to vary as its perpendicular distance from a plane touching the surface at the extremity of its axis.

45. To find the centre of gravity of a semi-circle, in which every particle is endued with a gravity proportional to its distance from the centre of the circle.

46. To find the centre of gravity of a hemisphere, in which every particle is endued with a gravity proportional to its distance from the centre of the sphere.

47. The magnitudes of two forces being given, to find the directions, in which they must be applied to the ends of the axis of a given heavy cylinder, so as to support the axis in a given position.

48. If two right cones, of the same uniform gravity, but of unequal altitudes, have equal bases, and rest each of them upon a prop, placed under any point in the circumference of its base, equal forces, applied at their summits, perpendicular to their axes, will retain their axes in a vertical position.

49. If a cone of a given weight, rest upon two given props, placed under the extremities of its axis, to find the pressure on each prop.

50. If two similar right cones rest each of them upon a prop, placed under a point in the circumference of its base, the forces, applied at their summits, perpendicularly to their axes, which keep the axes erect, are to one another as the weights of the cones.

51. If one extremity of a beam, which is a parallelepiped, rest upon a vertical post, to find the position of a cross-bar, of given length, to be fixt into the post, so as that it may best support the beam.

52. If a given heavy triangle is to be supported by three equal props, one applied at its vertex, and the other two in its base, so that its plane shall be horizontal, the place of one of these two being given, to find that of the other, so that they may sustain equal pressures.

53. If a given heavy cylinder, moveable about either extremity of its axis, be divided into n equal parts, their moving forces shall be as the successive odd numbers 1, 3, 5...., beginning from the pivot of rotation.

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54. To divide a given heavy cylinder, moveable about either extremity of its axis, into two parts, the moving forces of which shall be to one another in a given ratio.

55. To divide a given heavy cylinder, moveable about either extremity of its axis, into a given number of parts, the moving forces of which shall be equal to one another.

56. If a small motion be given to a system of weights, which is in a state of equilibrium, the perpendicular distance of the centre of gravity of the system, from a given horizontal plane, will remain the same as before: and conversely.

57. If a system of weights be in a state of equilibrium, the height of its centre of gravity, above a given horizontal plane, is the greatest, or the least, of those which obtain when the system loses its position of equilibrium.

58. If a given heavy body rest upon any two fixt surfaces, the sum of the pressures on the two surfaces, is always greater than the weight of the given body.

59. If a given heavy sphere be supported by two given inclined planes, and if any triangle be taken, of which the plane passes through the sphere's centre, and the two sides through the points of contact, and of which the base is horizontal, the base and the sides shall be respectively as the weight of the sphere, and the pressures on the two inclined planes.

60. If a heavy sphere rests on two given inclined planes, its pressures against each of them, in an horizontal direction, are equal to one another.

61. If two given heavy beams press against each other at their bases, on the ground plane, and also rest against two given semicircles, the diameters of which are in the same straight line, on the ground plane, to find the position of equilibrium.

62. To find the position of equilibrium of a heavy cylindrical rod, resting against two points of a hollow hemisphere, first, when the length is greater than a diameter of the sphere, secondly, when it is less than a diameter.

63. If a smaller sphere, of given weight and dimensions, be sustained, partly by resting against a given point of a sphere of greater given dimensions, and partly by a weight connected with

its centre, by means of a string passing over a given fixt pulley, to find the weight.

64. If a given heavy cube, pressing against each of two given immoveable spheres, be at rest in a given position, to compare, with its weight, the quantity of pressure against each of the spheres.

65. If two heavy spheres, of given magnitudes, be put into a perfectly smooth and hollow hemisphere, to find the distances of the points of contact from the vertex of the hemisphere, when there is an equilibrium.

66. To find a weight which shall sustain a given weight resting on a given point of the curve of a given parabola, the two weights being connected by means of a string passing over a fixt pully, which has its centre in the focus of the parabola.

67. If a given straight lever, loaded at a given point with a given weight, rest with one of its extremities against a given inclined plane, to find the nature of a curve, against any point of which, if the other extremity of the lever rest, there shall be an equilibrium.

68. A curve being given in a vertical plane, to find another curve in the same plane, such that two given weights, connected by a string of a given length, which passes over a fixt pulley, shall balance each other, resting upon any two points whatever of the two curves.

69. If a weight, of any given magnitude, however great, hangs freely, suspended by a string, any given power, however small, acting on the string in any direction, that is not vertical, will draw the string from the vertical line.

70. If the string, to which a given weight, hanging freely, is attached, be acted upon, at a given point, by a given force, in a given direction, to find the height, to which the given weight will thus be made to ascend.

71. If one extremity A of a string $ABCD$ be fixt at a given point A, if the string pass over a fixt pully at C, and if a given weight W be hung at the given point B, to find the weight, which must be hung at the extremity D, so that the part BC of the string may be horizontal, the point C being lower than the point A.

72. If a given heavy cone be suspended by a string, of given length, passing over a tack, and fastened to the two extremities of the cone's axis, to find the position of equilibrium, and the tension of the string.

73. If a triangle, in which the gravity is supposed to be uniform in any parallel to the base, whilst, in two different parallels, it varies as their perpendicular distances from the base, rest upon three equal props, placed under given points of the triangle, to find the relative pressure on each of them.

74. If a straight lever AB be loaded at one extremity A, with a given weight, at every point P, between A and B, with a weight, which is to the given weight as $\overline{PB}^2$ is to $\overline{AB}^2$, and at B with a weight equal to the sum of all the rest of the weights, to find the place of the fulcrum, so that there may be an equilibrium.

75. An horizontal straight line being given, to draw from either of its extremities a vertical, such that the time in which a heavy body falls down it, and afterwards, with its acquired velocity, describes the horizontal line, shall be a minimum.

76. To find the space due to the velocity acquired by a heavy body in falling down a system of planes, inclined at given angles to the horizon and to each other.

77. If two given inclined planes have a common altitude, to cut off from the one a part, which shall be described by a falling body in the same time as the other.

78. To find the locus of all the points, from each of which if two straight lines be drawn to two given points, the times of descent through them, by a body falling freely, shall be equal to one another.

79. The times in which two planes, of the same altitude, are described by two heavy bodies, projected down them with equal velocities, are to one another as the lengths of the planes.

80. If two given heavy bodies begin to descend from the same point, the one in a vertical line, the other in a line which has a given inclination to the horizon, to find the path described by their common centre of gravity.

81. If two given heavy bodies be projected, at the same instant, from the same given point, with given velocities, the one along an horizontal plane, the other perpendicularly to the horizon, to find the path described by their centre of gravity.

82. If a heavy body fall down a given inclined plane, to compare the spaces described, in equal and given times, at the beginning and at the end of its descent.

83. Two equal spaces being given in the line described by a body falling freely from rest, the one at the beginning of motion, the other between two given points of that line, to compare the times in which they are described.

84. From the bisection of a given vertical straight line, which rests on the ground, to draw, to the ground plane, a straight line, down which if a heavy body begin to fall, with the velocity due to half the given line, the time of its descent shall be equal to the time of falling freely through half the given line.

85. To find the perpendicular altitude, above a given inclined plane, through which a heavy body will be as long in falling from rest, as it is in describing the given inclined plane, beginning to fall down it with the velocity due to that perpendicular altitude.

86. If two straight lines, meeting in a point of the ground plane, be inclined at given angles to the horizon, to find a point in one of the lines, from which, if a heavy body fall down that line, a given point, in the other line, will be the limit of its ascent.

87. If two heavy bodies be projected, with equal velocities, from the same point, the one directly downwards, the other along a given inclined plane, to find the spaces described by them, in any the same given time.

88. If two heavy bodies be projected, with equal velocities, from the extremity of the vertical diameter of a vertical circle, the one down the diameter, the other along any chord terminated in that extremity, the chord shall be described in a less time than the diameter.

89. If a heavy body be projected along a given inclined plane, to assign two points in its course, between which it shall be as

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long in passing, as it would be in acquiring the velocity of projection, by falling freely from a state of rest.

90. If a heavy body, after descending freely through a given vertical line, begin to fall, with its acquired velocity, down an inclined plane, the part of the plane which is described in the same time as the given vertical line, is greater than the double and less than the triple of that given line.

91. If a heavy body be projected up a given inclined plane, with a velocity acquired in the time t, by falling freely through the given vertical line a, the space which it will describe on the plane in the time t, shall be greater than a.

92. Of all inclined planes, which have the same given altitude, to find that which shall be described, by a heavy body falling down it, in a given time.

93. To place an inclined plane, of a given length, so that it shall be described, in the given time t, by a heavy body, projected, first, down it, and secondly, up it, with a velocity acquired by the body falling freely for the time t: to find also the limitations of the problem.

94. From the greater of two given inclined planes, having the same altitude, to cut off a part, which shall be described by a falling body, first, towards the beginning of its motion; and secondly, towards the end of its motion, in the same time as that in which the lesser plane is described.

95. To determine the velocity, with which a heavy body must be projected down a given inclined plane, so as that it shall be exactly as long, in describing the plane, as it is in falling from rest through the altitude of the plane.

96. To determine a portion of a given inclined plane, equal in length to the plane's altitude, which shall be described, by a falling body, in the same time as that in which the altitude is described, by a body falling through it from rest.

97. If ABC be a triangle, the plane of which is vertical, and the base BC horizontal, to find a point D in BC, such that, AD being joined, the time of a heavy body's falling down AD

and afterwards describing *DC*, with the velocity thus acquired, shall be equal to the time in which it would fall down *BC*.

98. To compare the time in which a heavy body falls down the vertical diameter of a given circle, with the time in which two chords, drawn from its extremities to a given point in the circumference, are described by a body falling from rest down the upper chord, and entering upon the lower chord with the velocity thus acquired.

99. One of the two semi-diameters, which bound the quadrant of a given vertical circle, being vertical, and the centre being the highest point in it, to find a point in the arch, to which if two chords be drawn, they shall be described in equal times, by a heavy body falling from rest down the upper chord, and entering upon the lower with the velocity thus acquired.

100. One of the two semi-diameters, which bound the quadrant of a given vertical circle, being vertical, and the centre being the highest point in it, a heavy body shall be longer in falling from rest down the chord of the whole arch of the quadrant, than in falling down two chords drawn from its extremities to any given point of the arch; the body being supposed to enter on the lower chord with the whole of the velocity acquired in falling down the upper.

101. If a heavy body is to pass to a given point on the ground, by falling from a given point above it, down an inclined plane, and then moving on, with that part of its acquired velocity, which is not destroyed, by going from the one plane to the other, to find the position of the inclined plane, so that the whole time of the motion shall be a minimum.

102. If two spheres, the weights of which are W, w, and their diameters D, d, shot in a vertical direction into a bank of clay, penetrate to the depths H, h; to determine the ratio of the velocities of impact.

103. If a weight W, suspended from the lowest block of a system of $2n$ pullies, where the same string passes round all the pullies, and the parts of it between the pullies are parallel,

raise a weight X hung at the end of the string, the quantity of motion generated in X, in a given time, shall be a maximum, when $X : W :: \sqrt{2n+1}-1 : 4n^2$.

104. If a square, the plane of which is horizontal, and which has a circle inscribed in it, be moveable about one of its sides, to find three points in the circumference of the circle, at given distances from each other, from which if three equal weights be suspended, they shall exert the greatest force to turn the square about its axis of motion.

105. If a body begin to move along a given inclined plane, acted upon by a given centripetal force, tending to a given point below the plane, and varying as the distance from that point, to find how far, and how long, the body will move along the plane.

106. To find the nature of a curve, down which if a heavy body fall, the perpendicular pressure against the curve shall vary as the curve's vertical ordinate; the tangent of the curve being perpendicular to its base, at the beginning of the motion.

107. To find the nature of a curve, which shall be every where pressed with an equal centrifugal force, by a heavy body falling down it.

108. Of all parabolic arcs, in the same vertical plane, having a common vertex, and a common horizontal axis, to find that, along which if a heavy body begin to fall from rest, at the common vertex, it shall soonest reach a given straight line cutting all the parabolas.

109. If a body be projected, with a given velocity, from a given point, along an horizontal plane, in a given medium, where the resistance varies as the square of the velocity, to find its velocity, after it has described a given space, and the time in which the given space will have been described.

110. If a body be supposed to be projected with an infinitely great velocity, along an horizontal plane, in a given medium, where the resistance is varied as the square of the velocity, to find its velocity after it has described a given finite space.

111. If two heavy bodies be projected, at the same instant, with given velocities, from two given points of the same vertical straight line, the higher body downwards, the lower upwards, to find the place where they will meet, in a given medium, where the resistance is varied as the square of the velocity.

112. To find the velocity with which a heavy body must be projected directly downwards, in a medium where the resistance is proportional to the square of the velocity, so that its velocity may, at every point of its descent, be equal to the velocity of projection.

113. If in a medium, where the resistance varies as the square of the velocity, a heavy body fall, from rest, for an indefinite time, to find the greatest velocity, which it can acquire.

114. If a given triangle, suspended from its base, vibrate flatways in a medium, where the resistance is varied as the square of the velocity, to compare the resistances on any two lines in it, drawn parallel to the base, and terminated by the sides.

115. If an hyperbola, suspended from one of its assymptotes, vibrate flatways in a resisting medium, to compare the resistances on any two ordinates drawn to the other asymptote, first, when the resistance varies as the velocity; secondly, when it varies as the square of the velocity.

116. If a parabola, suspended from its base, vibrate flatways in a medium, where the resistance is varied as the n^{th} power of the distance, to find the ordinate which suffers the greatest resistance.

117. Each of the two persons, *A* and *B*, being supposed to throw a stone of given weight, with his utmost strength, from the top of a tower, of given height, in an horizontal direction, and the distances from the foot of the tower, at which the stones strike the ground plane, being measured, to find the ratio of the strength of *A* to the strength of *B*.

118. If two given heavy bodies be projected at the same instant, from the same point, and in the same direction, but with

different given velocities, to find the greatest height, to which their centre of gravity will ascend.

119. If two heavy bodies be projected towards each other, at the same instant, from two given points, so as to describe the same parabola, to find the point in which they will meet.

120. The length of an inclined plane being given, to find its inclination to the horizon, so that a body being projected up it with a given velocity, the range of the parabola described, after the body leaves the plane, may be a maximum.

121. If a heavy body fall down the curve of a semi-cycloid, of which the plane and the axis are vertical, to compare the times of its describing the two parts into which any given point divides the whole curve.

122. If the arc of a cycloid be divided into four equal parts, the time of a pendulum's motion through the first quarter, shall be equal to one-third of the whole time of oscillation.

123. If from any point M, in a semi-cycloid AMV, of which V is the vertex, a normal be drawn to the curve meeting the base in K, the time in which a heavy body falls down the curve from A to M, shall vary as AK.

124. If, from the ends of the chord of a given cycloidal arc, terminated one way at an extremity of the base, two straight lines be drawn, the one from the lower extremity a normal to the curve, and the other perpendicular to that normal, this perpendicular shall be to the chord, as the time in which a heavy body falls down the curve, is to the time in which it falls down the chord.

125. To determine an arch of a given vertical cycloid, having its base horizontal, through which, if a heavy body fall from the extremity of the base, the time of its motion shall be in a given ratio to the time of its falling down the chord of the arch: and to assign the limits of inequality in the given ratio.

126. If a heavy body fall down the curve of a semi-cycloid, having its plane and its axis vertical, to find the point in which the perpendicular pressure against the curve, is the greatest.

127. If a string of given length, and having its two extremities fixt to two points in the same horizontal line, be stretched

by a heavy ring, of inconsiderable magnitude, hanging on its middle and lowest point, and if the ring be made, by a force impressed on it horizontally, to run along the string, to find the time in which it will perform a small oscillation.

128. To find a point in a given finite straight line, through which, if an axis be drawn perpendicular to the line, the momentum of inertia of the line, referred to this axis, shall be one-third of the momentum of inertia of the line, referred to a parallel axis drawn through the end of the line.

129. To find the space described, in a given time, by a given heavy sphere, rolling down a given inclined plane.

130. To determine the ratio of the base to the altitude of a cone, so that when the cone, vibrates about an axis passing through its vertex, its centre of percussion shall be in the centre of its base.

131. If a wire, of indefinite length, and of inconsiderable weight, vibrate about an horizontal straight line, to find the place of a given heavy cylinder, with the axis of which the wire is made to coincide, so that the centre of oscillation may be at a given distance (39.2 inches) from the axis of suspension.

132. A heavy cone, the base of which is given, being suppose to make small oscillations about an axis passing through its vertex, to determine its altitude, so that the time of an oscillation may be a minimum.

133. A heavy paraboloid being given, to find a point in its axis, through which, if an horizontal wire be made to pass, the time of the solid's oscillation about it, through a small arc, shall be a minimum.

134. If a parallelepiped, of given weight and dimensions, hang suspended from an horizontal axis, which bisects its uppermost plane, and is at right angles to two of its opposite vertical planes, to find the point in which it must be struck, horizontally, by a ball equal to it in weight, moving with a given velocity, so that half the velocity of the ball shall be communicated to the point of impact.

135. Two wires of given weight, and of given and equal lengths, being supposed to meet at a given angle in the centre of a given heavy sphere, to which they are fastened, and their other extremities being attached to an horizontal axis, about which the pendulum is moveable, to find the time in which it will perform a small oscillation.

136. If the axis of a hemisphere, which rests with its vertex on an horizontal plane, be moved from its vertical position, through a given small angle, and be then suffered to oscillate, to compute the time of one of its oscillations.

137. If two perfectly hard bodies A and X, moving in opposite directions with the velocities a, b, meet one another, to find X, so that the quantity of motion communicated to it in the direction of A's motion may be a maximum; $A.a$ being supposed to be greater than $X.b$.

138. If A, B be two balls, of given magnitude and elasticity, to find an intermediate ball X, such that the velocity communicated, by impact, from A through X to C, may be equal to the velocity communicated from A to C, when A moving with the same velocity as before, immediately strikes C.

139. If a ball A, of given elasticity, fall from a given point in a vertical line, to find a lower point in the same line, from which if another equal ball B, of equal elasticity, be let fall, A shall meet B in the point to which B ascends, after having been reflected n times from the ground plane: to determine, also the point in which the bodies will next meet, after their first impact.

140. If a heavy and imperfectly elastic ball is projected vertically, with a given velocity, towards an horizontal and hard plane, at which it is reflected, and reaches the point of projection in a given time, to find the force of elasticity of the body, and the height of the point from which it is projected.

141. If an imperfectly elastic ball strike a given horizontal plane, in a direction making with a vertical line an angle, the cosine of which is the double of its sine, to determine the force of elasticity, so that the angle made by the direction of the reflected ball with the vertical, may be the double of the former angle.

142. If a perfectly elastic ball be projected from a given point, in the circumference of an horizontal circle, along the chord of an arch, which is to the whole circumference as m is to n, it will return to the given point, after $n - 1$ reflexions at the circumference: m and n being prime to each other.

143. Two indefinite planes being given, the one parallel to the horizon, and the other inclined to it, to find a point in the horizontal plane, from which, if a heavy and perfectly elastic ball be dropped, it shall be reflected by the inclined plane, so as to hit a given mark.

144. To find the inclination to the horizon of an indefinite plane, against a given point of which, if a heavy and perfectly elastic ball impinge with a given velocity, it shall be reflected so as to pass through another given point in that plane; the two given points being in the same perpendicular to the intersection of the inclined plane and the horizon.

145. To find that chord of a vertical circle, drawn from the upper extremity of its vertical diameter, down which if a ball of given elasticity descends, and is reflected by an horizontal plane, passing through the extremity of the chord, its range on that plane shall be a maximum.

146. If a perfectly elastic body be projected from a given point on the ground, at a given angle of elevation to the horizon, with a given velocity, to find the point in its trajectory where it must be reflected by a vertical plane, perpendicular to the plane of the trajectory, so as to hit a given mark on the ground: and to assign the limitations of the problem.

147. If the plane and axis of a given parabola be vertical, to find a point in the base from which if a heavy and perfectly elastic ball be dropped, it shall, after having been reflected by the curve, fall at the vertex.

148. In the direct impact of two bodies A, B, the elasticity of which is designated by p, if V' be the velocity lost by A, and v' the velocity gained by B, the whole loss of *vis viva* shall be expressed by $\frac{1-p}{1+p}(A \,.\, V'^2 + B \,.\, v'^2)$.

149. If a perfectly elastic ball *A* impinge directly upon an other ball *B*, at rest, in a medium, where the resistance varies as the square of the velocity, and if *B* is reflected by the plane which bounds the medium, to find the point in which the two balls will meet after impact; their magnitudes, and the distance of *B* from the side of the medium being given, and gravity being supposed not to act upon either of them.

150. To compute the height, above the bottom of a medium, where the resistance varies as the square of the velocity, from which a heavy and perfectly elastic body must be let fall, in the medium, so that when it has been reflected at the bottom, it may rise to a given point.

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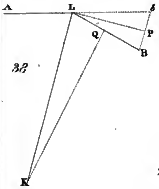

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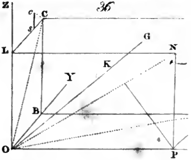

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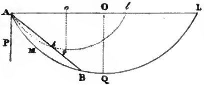

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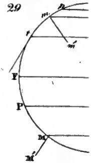

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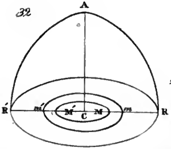

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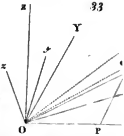

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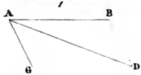

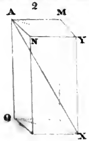

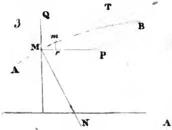

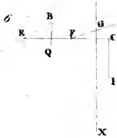

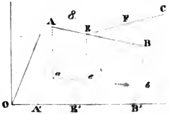

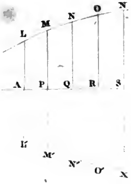

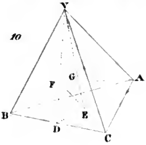

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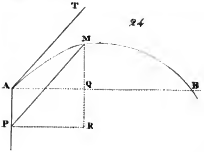

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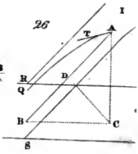

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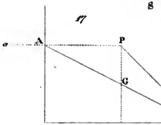

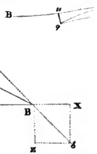

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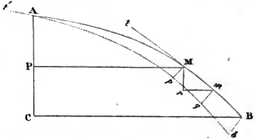

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